

PROBLEMS to CONSIDER
for/in relation to THE COURSE
"THE GAUGE PRINCIPLE"

Problem 1. Let $\chi_H \in \hat{H}^3(M)$ be a differential character of curvature $H \in \mathbb{Z}_{dR}^3(M)$ (see: [p.2.](#)). Using its defining property: $\chi_H \circ \partial_3 C = \exp(i \int_C H) \quad \forall C \in C_3(M)$ (a 3-chain), derive the contribution of the functional

$$C^\infty(\Sigma, M) \ni X \longmapsto \chi_H(X(\Sigma))$$

to the lūtd eqⁿs of the PQG σ -model (see: [q.1.](#)).

Problem 2. Consider a Poisson manifold (V, Π) given by a real vector space V (of say-dimension D) & a linear Poisson bivector $\Pi(\varphi) = C^{ab}_c \varphi^c \frac{\partial}{\partial \varphi^a} \wedge \frac{\partial}{\partial \varphi^b}$, written in the standard (global) coordinates φ^a on V associated with a basis $\{e_a\}_{a \in \overline{1, D}}$ of V , i.e., $\varphi^a(\varphi^b \triangleright e_b) = \varphi^a$. Prove that Π endows $V^* \cong \mathfrak{g}$ with the structure of a Lie algebra. Assume the latter quadratic, that is—equipped with a non-degenerate symmetric ad-invariant bilinear $\kappa_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$.

Write out the Poisson σ -model for the fields $X \in C^\infty(\Sigma, \mathbb{R}) \otimes \mathfrak{g}$ & $A \in \Omega^1(\Sigma) \otimes \mathfrak{g}$ in terms of $\kappa_{\mathfrak{g}}$ & the Lie bracket on $\mathfrak{g}$. ②

Derive the corresponding field e_2^{μ} & consider the infinitesimal gauge transformations from q.4. in the natural presentation of X & A adopted here. Propose an integrated version of these transformations & prove the invariance of the action functional under these integrated gauge transformations.

Problem 3. Show that bijections of the Atiyah groupoid $\text{At}(P)$ of a principal G -bundle $P \rightarrow \Sigma$ are in a one-to-one correspondence with G -equivariant automorphisms of P .

Problem 4. Prove $\text{Prop} \approx 3$.

Problem 5. Consider principal $U(1)$ -bundles $(P_a, \Sigma, U(1), \pi_{P_a}), a \in \{1, 2\}$ over the common base Σ . Using the Quotient Manifold Theorem prove that the quotient $(P_1 \times_{\pi_{P_1}} P_2)/G$ of the smooth manifold $P_1 \times_{\pi_{P_1}} P_2 = \{ (p_1, p_2) \mid p_a \in P_a, a \in \{1, 2\} \text{ and } \pi_{P_1}(p_1) = \pi_{P_2}(p_2) \}$ (π_{P_a} are surjective submersions!) is — again — smooth.

Show that the quotient is naturally a principal $U(1)$ -bundle over Σ — we denote it as $P_1 \otimes P_2$.

Given an open cover $\{\mathcal{O}_i\}_{i \in I}$ over which both P_1 and P_2 trivialize as $\tau_i^{(a)}: \pi_{P_a}^{-1}(\mathcal{O}_i) \xrightarrow{\sim} \mathcal{O}_i \times G, i \in I$ ($U(1)$ -equivariantly),

with the respective transition 1-cocycles $g_{ij}^{(a)}: \mathcal{O}_i \rightarrow U(1)$, find the induced trivializations & the transition cocycle of $P_1 \otimes P_2$. (5)

Problem 6.: Consider a metric manifold (M, g_M) with a trivial
 gerbe $(B, 0, 1)$ over it. Assume given a group $G \subset \text{Isom}(M, g_M)$
 such that $B \in \Omega^2(M)^G$ (G -invariant). Gauge the symmetry $G \curvearrowright M$
 of the PAg σ -model through 'minimal coupling' ($\equiv$ verticalisation)
 for a trivial principal bundle $\Sigma \times G$ with trivial connection

$$\Theta(\sigma, g) = \text{id}_{T_\sigma G} + T_\sigma \tau_g \circ A(\sigma) \quad (\tau_g: G \ni h \mapsto h \cdot g, g \in G)$$
 over Σ with η_Σ Lorentzian, & write
 the resulting action functional as a polynomial in the (here global)
 gauge field $A \in \Omega^1(\Sigma) \otimes \text{Lie } G$. Treating the latter as a field
 of the new theory, derive its Euler-Lagrange eqⁿs.
 Assume the action of G on M free & proper, & such
 that g_M defines a positive definite metric on fibres

of the ensuing principal G -bundle $M \rightarrow M/G (\in \text{Man})$. Using that assumption, solve the equations for A , & substitute the solution back in the action functional. Interpret the result as a new σ -model for $C^\infty(\Sigma, M)$ with tensors $(\tilde{g}_M, \tilde{B})$. Prove that the latter are pullbacks of tensors from M/G , thus demonstrating the direct descent of dynamics to the symmetry orbit space M/G as the end result of the gauging.

Problem 7. We identify ISOMORPHISM CLASSES of gerbes with cohomology classes $[(B_i, A_{ij}, g_{ijk})]$ of BD 2-cocycles $(B_i, A_{ij}, g_{ijk}) \in \check{C}^0(\underline{\mathcal{Q}}^2) \oplus \check{C}^1(\underline{\mathcal{Q}}^1) \oplus \check{C}^2(\underline{\mathcal{Q}}^0)$. On classes, we have the product

$$[(B_i, A_{ij}, g_{ijk})] \otimes [(B'_i, A'_{ij}, g'_{ijk})] := [(B_i, A_{ij}, g_{ijk}) \overset{\text{as in the lecture}}{\oplus} (B'_i, A'_{ij}, g'_{ijk})] = [(B_i + B'_i, A_{ij} + A'_{ij}, g_{ijk} g'_{ijk})]$$

Denote $\mathcal{W}^3(M; H) := \{ [(B_i, A_{ij}, g_{ijk})] \mid dB_i = 0 \ \forall i \in I \}$ (IS-CLASSES of gerbes of curvature $H \in \mathbb{Z}_{\text{dr}}^3(M)$). Assume $\mathcal{O} = \{\mathcal{O}_i\}_{i \in I}$ good & show

$$\mathcal{W}^3(M; \mathcal{O}) \xrightarrow[\text{red}]{\simeq} \check{H}^2(M, U(1)),$$

where on the right-hand side there appears the cohomology of the complex

$$\dots \xrightarrow{\check{\Delta}_{(p-1)}} \check{C}^p(\mathcal{O}, U(1)) \xrightarrow{\check{\Delta}_{(p)}} \dots,$$

for $\check{\Delta}_{(p)}$ as in the lecture, but restricted to locally constant maps valued in $U(1)$, $\check{C}^p(\mathcal{O}, U(1)) \equiv \check{C}^p(\underline{\mathcal{Q}}^0) \cap \ker d_{(0)}$. ⑧

Prove that $\mathcal{W}^3(M; H)$ is a torus under the action

$$\mathcal{W}^3(M; H) \times \check{H}^2(M, U(1)) \longrightarrow \mathcal{W}^3(M; H), ([B, A_{ij}, g_{ijk}], [c_{ijk}]) \longmapsto [B, A_{ij}, g_{ijk}] \otimes \iota^{-1}([c_{ijk}]).$$