Compact quantum groups - lecture 4

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Example: dual of a discrete group

Let Γ be a discrete group and consider the left regular representation λ of Γ on the Hilbert space $\ell^2(\Gamma)$:

$$(\lambda_p\psi)(q) = \psi(p^{-1}q), \qquad \psi \in \ell^2(\Gamma),\ p, q \in \Gamma.$$

Furthermore let $C^*_r(\Gamma)$ be the C*-algebra generated by the operators $\{\lambda_p | p \in \Gamma\}$ (i.e. the smallest closed subalgebra of $B(\ell^2(\Gamma))$ containing these operators which happens to coincide with the closure of the span of $\{\lambda_p | p \in \Gamma\}$).

Next consider the unitary operator W on $\ell^2(\Gamma) \otimes \ell^2(\Gamma) \cong \ell^2(\Gamma \times \gamma)$ defined by

$$(W\Psi)(p, q) = \Psi(p, q^{-1}p), \qquad \Psi \in \ell^2(\Gamma \times \Gamma),\ p, q, \in \Gamma.$$

Then we have $(W^*\Psi)(x, y) = \Psi(x, xy^{-1})$ and consequently

$$\begin{aligned}(W(\lambda_p \otimes \mathbb{1})W^*\Psi)(a, b) &= ((\lambda_p \otimes \mathbb{1})W^*\Psi)(a, b^{-1}a) \\ &= (W^*\Psi)(p^{-1}a, b^{-1}a) = \Psi(p^{-1}a, p^{-1}b) = ((\lambda_p \otimes \lambda_p)\Psi)(a, b).\end{aligned}$$

Example: dual of a discrete group

It follows that the map $x \mapsto W(\mathbb{1} \otimes x)W^*$ restricted to $C_r^*(\Gamma)$ has range in $C_r^*(\Gamma) \otimes C_r^*(\Gamma)$ and maps each λ_p to $\lambda_p \otimes \lambda_p$. Defining Δ to be this mapping we easily see that Δ is a coassociative unital $*$-homomorphism. Moreover for any $p, q, a, b \in \Gamma$

$$\Delta(\lambda_p)(\mathbb{1} \otimes \lambda_{p^{-1}q}) = \lambda_p \otimes \lambda_q \quad \text{and} \quad (\lambda_{ab^{-1}} \otimes \mathbb{1})\Delta(\lambda_b) = \lambda_a \otimes \lambda_b$$

and such operators span a dense subspace of $C_r^*(\Gamma) \otimes C_r^*(\Gamma)$. It follows that the density conditions from the definition of a compact quantum group are satisfied, i.e. the pair $(C_r^*(\Gamma), \Delta)$ describes a compact quantum group. This compact quantum group is usually denoted by the symbol $\widehat{\Gamma}$ and it is referred to as the **dual** of Γ or, more precisely, the **reduced dual** of Γ.

Example: dual of a discrete group II

Apart from $C^*_r(\Gamma)$ one can also form the **universal** or **full group** C^***-algebra** $C^*(\Gamma)$ of Γ. It is the C^*-algebra generated by symbols $\{u_p\}_{p\in\Gamma}$ which are all unitary and satisfy the relations $u_p u_q = u_{pq}$ for all $p, q \in \Gamma$. The existence of the comultiplication $\Delta\colon C^*(\Gamma) \to C^*(\Gamma) \otimes C^*(\Gamma)$ is an easy consequence of the universal property of this C^*-algebra and it is equally easy to see that $(C^*(\Gamma), \Delta)$ describes a compact quantum group.

Since the operators $\{\lambda_p\}_{p\in\Gamma}$ of the left regular representation are unitary and satisfy the appropriate relations, there exists a unique (quotient) map $\pi\colon C^*(\Gamma) \to C^*_r(\Gamma)$ which obviously satisfies $(\pi \otimes \pi) \circ \Delta = \Delta \circ \pi$ (these are comultiplications on two possibly different C^*-algebras). It is known that π is an isomorphism iff the group Γ is amenable. The compact quantum group in this example is often called the **universal dual** of Γ.

Theorem (S.L. Woronowicz)

Let A be a unital C*-algebra generated by elements $\{u_{i,j}\}_{i,j=1,\dots,n}$ such that the matrices

$$u = \begin{bmatrix} u_{1,1} & \cdots & u_{1,n} \\ \vdots & \ddots & \vdots \\ u_{n,1} & \cdots & u_{n,n} \end{bmatrix} \quad \text{and} \quad u^{\mathrm{t}} = \begin{bmatrix} u_{1,1} & \cdots & u_{n,1} \\ \vdots & \ddots & \vdots \\ u_{1,n} & \cdots & u_{n,n} \end{bmatrix}$$

are invertible in $\mathrm{Mat}_n(A)$ and assume there exists a unital $*$-homomorphism $\Delta\colon A \to A \otimes A$ such that

$$\Delta(u_{i,j}) = \sum_{k=1}^{n} u_{i,k} \otimes u_{k,j}.$$

Then (A, Δ) describes a compact quantum group.

Proof

It is easy to see that Δ is coassociative, since it is enough to check this on generators of A.

To see that $\Delta(A)(\mathbb{1}\otimes A)$ is dense in $A\otimes A$ it is enough to check that the set

$$B=\left\{a\in A\,\middle|\,\exists\, x_1,\dots,x_m,y_1,\dots,y_m\in A\quad a\otimes\mathbb{1}=\sum_{i=1}^{m}\Delta(x_i)(\mathbb{1}\otimes y_i)\right\}$$

is dense in A (indeed $\{b\otimes a|a\in A,\ b\in B\}$ is contained in $\Delta(A)(\mathbb{1}\otimes A)$ and the former set is linearly dense in $A\otimes A$).

The set B is a subalgebra of A because it is obviously a vector subspace and if

$$a\otimes\mathbb{1}=\sum_{i=1}^{m}\Delta(x_i)(\mathbb{1}\otimes y_i)\quad\text{and}\quad b\otimes\mathbb{1}=\sum_{j=1}^{m'}\Delta(x'_j)(\mathbb{1}\otimes y'_j)$$

then

$$ab\otimes\mathbb{1}=\sum_i\Delta(x_i)(\mathbb{1}\otimes y_i)(b\otimes\mathbb{1})=\sum_i\Delta(x_i)(b\otimes\mathbb{1})(\mathbb{1}\otimes y_i)=\sum_{i,j}\Delta(x_i)\Delta(x'_j)(\mathbb{1}\otimes y'_j)(\mathbb{1}\otimes y_i)=\sum_{i,j}\Delta(x_ix'_j)(\mathbb{1}\otimes y'_jy_i).$$

Proof

Now note that the assumption on Δ says that

$$\begin{bmatrix} \Delta(u_{1,1}) & \cdots & \Delta(u_{1,n}) \\ \vdots & \ddots & \vdots \\ \Delta(u_{n,1}) & \cdots & \Delta(u_{n,n}) \end{bmatrix} = \begin{bmatrix} u_{1,1} \otimes \mathbb{1} & \cdots & u_{1,n} \otimes \mathbb{1} \\ \vdots & \ddots & \vdots \\ u_{n,1} \otimes \mathbb{1} & \cdots & u_{n,n} \otimes \mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} \otimes u_{1,1} & \cdots & \mathbb{1} \otimes u_{1,n} \\ \vdots & \ddots & \vdots \\ \mathbb{1} \otimes u_{n,1} & \cdots & \mathbb{1} \otimes u_{n,n} \end{bmatrix}$$

so if

$$v = \begin{bmatrix} v_{1,1} & \cdots & v_{1,n} \\ \vdots & \ddots & \vdots \\ v_{n,1} & \cdots & v_{n,n} \end{bmatrix}$$

is the inverse of u then

$$\begin{bmatrix} \Delta(u_{1,1}) & \cdots & \Delta(u_{1,n}) \\ \vdots & \ddots & \vdots \\ \Delta(u_{n,1}) & \cdots & \Delta(u_{n,n}) \end{bmatrix} \begin{bmatrix} \mathbb{1} \otimes v_{1,1} & \cdots & \mathbb{1} \otimes v_{1,n} \\ \vdots & \ddots & \vdots \\ \mathbb{1} \otimes v_{n,1} & \cdots & \mathbb{1} \otimes v_{n,n} \end{bmatrix} = \begin{bmatrix} u_{1,1} \otimes \mathbb{1} & \cdots & u_{1,n} \otimes \mathbb{1} \\ \vdots & \ddots & \vdots \\ u_{n,1} \otimes \mathbb{1} & \cdots & u_{n,n} \otimes \mathbb{1} \end{bmatrix}.$$

Proof

The last equality means that each generator $u_{i,j}$ belongs to B.

Next we note that the assumed invertibility of u^{t} is equivalent to invertibility of

$$\overline{u} = \begin{bmatrix} u_{1,1}^* & \cdots & u_{1,n}^* \\ \vdots & \ddots & \vdots \\ u_{n,1}^* & \cdots & u_{n,n}^* \end{bmatrix}$$

(because $\overline{u} = (u^{\mathrm{t}})^*$), so multiplying the equality

$$\begin{bmatrix} \Delta(u_{1,1}^*) & \cdots & \Delta(u_{1,n}^*) \\ \vdots & \ddots & \vdots \\ \Delta(u_{n,1}^*) & \cdots & \Delta(u_{n,n}^*) \end{bmatrix} = \begin{bmatrix} u_{1,1}^* \otimes \mathbb{1} & \cdots & u_{1,n}^* \otimes \mathbb{1} \\ \vdots & \ddots & \vdots \\ u_{n,1}^* \otimes \mathbb{1} & \cdots & u_{n,n}^* \otimes \mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} \otimes u_{1,1}^* & \cdots & \mathbb{1} \otimes u_{1,n}^* \\ \vdots & \ddots & \vdots \\ \mathbb{1} \otimes u_{n,1}^* & \cdots & \mathbb{1} \otimes u_{n,n}^* \end{bmatrix}$$

from the right by the matrix of elements $\mathbb{1} \otimes w_{k,l}$, where $\{w_{k,l}\}$ are elements of $\overline{u}^{-1}$ we find that each $u_{i,j}^*$ also belongs to B.

Proof

Since B is an algebra containing all generators of A and their adjoints, it is dense in A.

In a similar way we show that $(A \otimes \mathbb{1})\Delta(A)$ is dense in $A \otimes A$ thus showing that (A, Δ) describes a compact quantum group. □

Definition

A compact quantum group $\mathbb{G}$ described by (A, Δ) as in the last theorem is called a **compact matrix quantum group**.

- ⇝ The original terminology referred to compact matrix quantum groups as **compact matrix pseudogroups**.
- ⇝ A compact matrix quantum group $\mathbb{G}$ with commutative $\mathrm{C}(\mathbb{G})$ is a (classical) compact group of matrices.

Example: the quantum group $\mathsf{SU}_q(2)$

Fix $q \in [-1, 1] \setminus \{-0\}$ and let $\mathrm{C}(\mathsf{SU}_q(2))$ be the C^*-algebra generated by α and γ such that

$$\alpha^*\alpha + \gamma^*\gamma = \mathbb{1}, \quad \alpha\alpha^* + q^2\gamma\gamma^* = \mathbb{1},$$
$$\alpha\gamma = q\gamma\alpha, \qquad \alpha\gamma^* = q\gamma^*\alpha,$$
$$\gamma^*\gamma = \gamma\gamma^*$$

(let us note that in the case of C^*-algebras the relation $\alpha\gamma^* = q\gamma^*\alpha$ actually follows from the remaining ones). Then there exists $\Delta\colon \mathrm{C}(\mathsf{SU}_q(2)) \to \mathrm{C}(\mathsf{SU}_q(2)) \otimes \mathrm{C}(\mathsf{SU}_q(2))$ such that

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

The pair $(\mathrm{C}(\mathsf{SU}_q(2)), \Delta)$ describes a compact matrix quantum group denoted by $\mathsf{SU}_q(2)$. For $q = 1$ this is the classical group $\mathsf{SU}(2)$.

Example: the quantum group $\mathsf{SU}_q(2)$

The fact that $\mathsf{SU}_q(2)$ is indeed a compact quantum group can be argued in many different ways. For example it follows from the fact that the algebra $\mathscr{A}_q$ discussed in the first lecture is a Hopf algebra. However a more direct proof can be obtained as follows: let

$$u = \begin{bmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{bmatrix}$$

and let $\lambda \in \mathbb{C}$ be such that $\lambda^2 = q$. Then

$$u^{\mathrm{t}} = \begin{bmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{bmatrix} u^* \begin{bmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{bmatrix}^{-1},$$

so u^{t} is invertible.

Example: The quantum group S_n^+

Let $\mathrm{C}(\mathsf{S}_n^+)$ be the C^*-algebra generated by elements $\{u_{i,j} \,|\, i,j = 1,\dots,n\}$ such that $u_{i,j} = u_{i,j}^2 = u_{i,j}^*$ for all i,j and

- for any i we have $\sum_j u_{i,j} = \mathbb{1}$,
- for any j we have $\sum_i u_{i,j} = \mathbb{1}$.

Note that these conditions mean that the matrix u with entries $u_{i,j}$ is unitary and so is its transpose.

The comultiplication $\Delta\colon\ \mathrm{C}(\mathsf{S}_n^+) \to \mathrm{C}(\mathsf{S}_n^+) \otimes \mathrm{C}(\mathsf{S}_n^+)$ is defined in the usual way:

$$\Delta(u_{i,j}) = \sum_{k=1}^{n} u_{i,k} \otimes u_{k,j}, \qquad i,j = 1,\dots,n.$$

Thus the pair $(\mathrm{C}(\mathsf{S}_n^+), \Delta)$ describes a compact quantum group S_n^+ called the **quantum permutation group**.

- ⇝ If we additionally impose the requirement that the generators $\{u_{i,j}\}$ of $C(S_n^+)$ pairwise commute, the resulting commutative algebra is isomorphic to the algebra of all continuous functions on the set of $n \times n$ matrices with $\{0, 1\}$ entries whose every row and every column sum to 1.
- ⇝ In other words the abelianization of $C(S_n^+)$ yields the algebra of functions on the set of permutation matrices which can be identified with the symmetric group S_n.
- ⇝ Thus S_n is a quantum subgroup of S_n^+.
- ⇝ In fact for $n = 1, 2, 3$ the algebra $C(S_n^+)$ is already commutative and thus isomorphic to $C(S_n)$.
- ⇝ One can show that the algebra $C(S_n^+)$ is not commutative and infinite-dimensional for $n \geq 4$.