Compact quantum groups - lecture 3

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Fall semester 2026/2027

- An element $a \in A$ is said to be **positive** if $a = c^*c$ for some $c \in A$.
- The set of positive elements of A is a cone denoted by A_+.
- Every element $x \in A$ is a linear combination of (at most) four positive elements.
- Let A and B be C*-algebras. A linear map $\Phi\colon A \to B$ is **positive** if $\Phi(A_+) \subset B_+$.
- A positive map is automatically continuous and satisfies $\|\Phi\| = \|\Phi(\mathbb{1})\|$ is A if unital.
- A **state** on A is positive functional ω on A with $\|\omega\| = 1$.

Theorem (Riesz)

Let X be a compact space and let ω be a state on $\mathrm{C}(X)$. Then there exists a unique Radon probability measure μ on X such that

$$\forall f \in \mathrm{C}(X) \quad \omega(f) = \int_X f \,\mathrm{d}\mu.$$

- Any $\varphi \in A^*$ can be expressed as a linear combination of four states.

Definition

A **compact quantum space** is an object of the category dual to the category of unital C*-algebras.

- ⇝ Thus to every compact quantum space $\mathbb{X}$ corresponds a unique unital C*-algebra A (and vice versa).
- ⇝ We often denote the C*-algebra corresponding to $\mathbb{X}$ by the symbol $\mathrm{C}(\mathbb{X})$ – this is a textbook example of "abuse of notation".
- ⇝ Mappings between compact quantum spaces $\mathbb{X}$ and $\mathbb{Y}$ are encoded in the (unital) $*$-homomorphisms $\mathrm{C}(\mathbb{Y}) \to \mathrm{C}(\mathbb{X})$.
- ⇝ The C*-algebra $\mathrm{C}(\mathbb{X}) \otimes \mathrm{C}(\mathbb{Y})$ is regarded as corresponding to $\mathbb{X} \times \mathbb{Y}$.
- ⇝ States on $\mathrm{C}(\mathbb{X})$ play the role of probability measures on $\mathbb{X}$.
- ⇝ By Gelfand's theorem a commutative C*-algebra A corresponds to a **classical** space X, i.e. we have $A \cong \mathrm{C}(X)$.

Proposition (Schwarz inequality for states)

Let A be C*-algebra and let ω be a positive functional on A. Then for any $a, b \in A$ we have

$$\left|\omega(a^*b)\right| \leq \omega(a^*a)^{1/2}\omega(b^*b)^{1/2}.$$

$\rightsquigarrow$ In particular if $x \in A$ satisfies $\omega(x^*x) = 0$ then $\omega(x^*y) = 0$ as well as $\omega(y^*x) = 0$ for all $y \in A$.

- A **representation** of a C*-algebra A on a Hilbert space $\mathscr{H}$ is a $*$-homomorphism $\pi\colon A \to \mathrm{B}(\mathscr{H})$.
- We will mostly consider unital C*-algebras and therefore tacitly assume that representations are unital.
- If π is a representation of A on $\mathscr{H}$ and $\Omega \in \mathscr{H}$ then the mapping

$$\omega\colon A \ni a \longmapsto \langle \Omega | \pi(a)\Omega \rangle \in \mathbb{C}$$

  is a positive functional on A. It is a state iff $\|\Omega\| = 1$.
- We say that a vector $\Omega \in \mathscr{H}$ is **cyclic** for π if the subspace $\{\pi(a)\Omega | a \in A\}$ is dense in $\mathscr{H}$.

Theorem

Let A be a C*-algebra and let ω be a positive functional on A. Then there exists a triple $(\mathscr{H}_\omega, \pi_\omega, \Omega_\omega)$, where

- $\mathscr{H}_\omega$ is a Hilbert space,
- π_ω is a representation of A on $\mathscr{H}_\omega$,
- Ω_ω is a cyclic vector for π_ω

such that

$$\forall\, a \in A \quad \omega(a) = \langle \Omega_\omega | \pi_\omega(a) \Omega_\omega \rangle.$$

Furthermore if $(\mathscr{H}', \pi', \Omega')$ is a triple of analogous objects with the same properties then there exists a unique unitary operator $U\colon \mathscr{H}_\omega \to \mathscr{H}'$ such that $\pi'(a) = U\pi_\omega(a)U^*$ for all $a \in A$.

Proof

Let

$$\mathcal{J}_\omega = \{a \in A \,|\, \omega(a^*a) = 0\}.$$

Let us see first that $\mathcal{J}_\omega$ is a left ideal of A. First of all $\mathcal{J}_\omega$ is stable under multiplication by scalars. Secondly, if $x, y \in \mathcal{J}_\omega$ then since

$$(x+y)^*(x+y) = x^*x + x^*y + y^*x + y^*y \leq 2(x^*x + y^*y)$$

(because $0 \leq (x-y)^*(x-y)$ implies that $x^*y + y^*x \leq x^*x + y^*y$) we have

$$0 \leq \omega((x+y)^*(x+y)) \leq 2(\omega(x^*x) + \omega(y^*y)) = 0,$$

so $x + y \in \mathcal{J}_\omega$.

Proof

Furhtermore, if $a \in \mathcal{J}_\omega$ and $b \in A$ then since $b^*b \leq \|b\|^2 \mathbb{1}$ and inequalities are preserved by the operation $x \mapsto a^*xa$ and ω is positive, we have

$$0 \leq \omega((ba)^*(ba)) = \omega(a^*b^*ba) \leq \|b\|^2 \omega(a^*a) = 0,$$

so $ba \in \mathcal{J}_\omega$.

Now if $a, b \in A$ and $x, y \in \mathcal{J}_\omega$ then the value $\omega((a+x)^*(b+y))$ does not depend on x and y:

$$\omega((a+x)^*(b+y)) = \omega(a^*b) + \omega(a^*y) + \omega(x^*b) + \omega(x^*y) = \omega(a^*b) + \omega(a^*y) + \overline{\omega(b^*x)} + \omega(x^*y)$$

and by Schwarz inequality $\omega(a^*y) = \omega(b^*x) = \omega(x^*y) = 0$.

Proof

In other words there is a well defined sesqiulinear form $\langle \cdot | \cdot \rangle$ on the vector space $A/\mathcal{J}_\omega$ given by

$$\langle a + \mathcal{J}_\omega | b + \mathcal{J}_\omega \rangle = \omega(a^* b).$$

It is easy to see that this form is strictly positive.

We let $\mathscr{H}_\omega$ be the Hilbert space obtained as the completion of $A/\mathcal{J}_\omega$ in the norm defined by the scalar product $\langle \cdot | \cdot \rangle$, so that, in particular, $A/\mathcal{J}_\omega$ is a dense subspace of $\mathscr{H}_\omega$.

Since $\mathcal{J}_\omega$ is a left ideal, the quotient space $A/\mathcal{J}_\omega$ is a left A-module, so every $a \in A$ defines a linear map $b + \mathcal{J}_\omega \mapsto ab + \mathcal{J}_\omega$. As

$$\|ab + \mathcal{J}_\omega\|^2 = \omega((ab)^*(ab)) \leq \|a\|^2 \omega(b^* b) = \|a\|^2 \|b + \mathcal{J}_\omega\|^2,$$

each such mapping extends to a bounded operator $\pi_\omega(a)$ on $\mathscr{H}_\omega$. It is easy to see that $a \mapsto \pi_\omega(a)$ is a unital homomorphism $A \to \mathrm{B}(\mathscr{H}_\omega)$.

Proof

Denote by Ω_ω the element $\mathbb{1} + \mathcal{J}_\omega \in A/\mathcal{J}_\omega \subset \mathscr{H}_\omega$. Then clearly

$$\{\pi_\omega(a)\Omega_\omega \,|\, a \in A\}$$

coincides with the dense subspace $A/\mathcal{J}_\omega \subset \mathscr{H}_\omega$. Now for any two vectors of the form $\pi_\omega(x)\Omega_\omega$ and $\pi_\omega(y)\Omega_\omega$ and any $a \in A$ we have

$$\langle \pi_\omega(x)\Omega_\omega \,|\, \pi_\omega(a)\pi_\omega(y)\Omega_\omega\rangle = \omega(x^*ay) = \omega((ax)^*y) = \langle \pi_\omega(a^*)\pi_\omega(x)\Omega_\omega \,|\, \pi_\omega(y)\Omega_\omega\rangle.$$

By continuity this means that

$$\langle \xi \,|\, \pi_\omega(a)\eta\rangle = \langle \pi_\omega(a^*)\xi \,|\, \eta\rangle$$

for all $\xi, \eta \in \mathscr{H}_\omega$ which shows that π_ω is a $*$-homomorphism. In other words π_ω is a representation of A on the Hilbert space $\mathscr{H}_\omega$ and, as we already established above, Ω_ω is a cyclic vector for π_ω.

Proof

By construction for any $a \in A$ we have

$$\langle \Omega_\omega | \pi_\omega(a)\Omega_\omega \rangle = \omega(\mathbb{1}^* a \mathbb{1}) = \omega(a).$$

It remains to prove the uniqueness part of the theorem.

Let $(\mathscr{H}', \pi', \Omega')$ be a triple of a Hilbert space, representation of A on $\mathscr{H}'$, and a cyclic vector such that $\langle \Omega' | \pi'(a)\Omega' \rangle = \omega(a)$ for all $a \in A$. Define the map U_0 form the dense subspace $A/\mathcal{J}_\omega$ of $\mathscr{H}_\omega$ to $\mathscr{H}'$ by

$$U_0(a + \mathcal{J}_\omega) = \pi'(a)\Omega'.$$

It is easy to check that this map is well-defined, since for $y \in \mathcal{J}_\omega$ we have

$$\left\| \pi'(y)\Omega' \right\|^2 = \omega(y^* y) = 0.$$

Proof

Furthermore U_0 is isometric

$$\left\|U_0(a+\mathcal{J}_\omega)\right\|^2 = \left\|\pi'(a)\Omega'\right\|^2 = \omega(a^*a) = \|a+\mathcal{J}_\omega\|^2$$

and has the dense subspace $\pi'(A)\Omega'$ as its range. Consequently U_0 extends to a unitary operator $U\colon \mathscr{H}_\omega \to \mathscr{H}'$ and for $a \in A$ and $\xi \in A/\mathcal{J}_\omega \subset \mathscr{H}_\omega$

$$U\pi_\omega(a)\xi = \pi'(a)U\xi$$

which means that $\pi'(a) = U\pi_\omega(a)U^*$. In particular, for $a = \mathbb{1}$ and $\xi = \Omega_\omega$, we obtain $U\Omega_\omega = \Omega'$. These properties determine U uniquely because

$$U(\pi_\omega(a)\Omega_\omega) = U\pi_\omega(a)U^*\Omega' = \pi'(a)\Omega'$$

dictates the values of U on a dense subspace of $\mathscr{H}_\omega$. □

- The construction presented in the theorem above is the **Gelfand-Naimark-Segal** or the **GNS construction**.
- A positive functional ω on a C*-algebra A is **faithful** if for any $a \in A$ the condition $\omega(a^*a) = 0$ implies $a = 0$.
- Clearly if ω is a faithful positive functional then the sesquilinear form $(a, b) \mapsto \omega(a^*b)$ is a scalar product on A (in other words $\mathcal{J}_\omega = \{0\}$, see the proof above).

Definition

A **compact quantum group** $\mathbb{G}$ is described by a pair (A, Δ) consisting of a unital cst-algebra and a unital $*$-homomorphisms $\Delta\colon A \to A \otimes A$ such that

1. $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$,
2. the sets

$$\Delta(A)(\mathbb{1} \otimes A) = \mathrm{span}\{\Delta(a)(\mathbb{1} \otimes b) \,|\, a, b \in A\} \quad \text{and} \quad (A \otimes \mathbb{1})\Delta(A) = \mathrm{span}\{(a \otimes \mathbb{1})\Delta(b) \,|\, a, b \in A\}$$

are dense in $A \otimes A$.

$\rightsquigarrow$ The C*-algebra A as above is usually denoted by the symbol $\mathrm{C}(\mathbb{G})$.

Example: classical compact group

Let G be a compact group. Then $A = \mathrm{C}(G)$ and $\Delta\colon\ \mathrm{C}(G) \to \mathrm{C}(G) \otimes \mathrm{C}(G) = \mathrm{C}(G \times G)$ defined by

$$\Delta(f)(p, g) = f(pq), \qquad f \in \mathrm{C}(G),\ p, q \in G$$

describes a compact quantum group.

Indeed, for $f, g, \in \mathrm{C}(G)$ the function $\Delta(f)(\mathbb{1} \otimes g)$ is

$$(p, q) \longmapsto f(pq)g(q) = (f \otimes g)(pq, q).$$

From the density of $\mathrm{C}(G) \otimes_{\mathrm{alg}} \mathrm{C}(G)$ in $\mathrm{C}(G \times G)$ we infer that the closure of $\Delta(\mathrm{C}(G))(\mathbb{1} \otimes \mathrm{C}(G))$ is the set of all functions of the form

$$(p, q) \longmapsto F(pq, q)$$

with $F \in \mathrm{C}(G \times G)$.

Example: classical compact group

But $(p,q) \mapsto (pq,q)$ is a homeomorphism $G\times G \to G\times G$, so this set is all of $\mathrm{C}(G\times G) = \mathrm{C}(G)\otimes\mathrm{C}(G)$. Similarly the closure of $(\mathrm{C}(G)\otimes \mathbb{1})\Delta(\mathrm{C}(G))$ is the set of all functions

$$(p,q) \longmapsto F(p,pq)$$

with $F \in \mathrm{C}(G\times G)$ which is easily seen to be all of $\mathrm{C}(G\times G)$ as well.

Remark

The mappings

$$\mathrm{C}(G) \otimes_{\mathrm{alg}} \mathrm{C}(G) \ni (f \otimes g) \longmapsto \Delta(f)(\mathbb{1} \otimes g) \in \mathrm{C}(G \times G),$$
$$\mathrm{C}(G) \otimes_{\mathrm{alg}} \mathrm{C}(G) \ni (f \otimes g) \longmapsto (f \otimes \mathbb{1})\Delta(g) \in \mathrm{C}(G \times G)$$

extend to unital $*$-homomorphisms $\mathrm{C}(G \times G) \to \mathrm{C}(G \times G)$ which correspond to the continuous maps

$$G \times G \ni (p, q) \longmapsto (pq, q) \in G \times G,$$
$$G \times G \ni (p, q) \longmapsto (p, pq) \in G \times G.$$

The fact that either of the maps $\mathrm{C}(G \times G) \to \mathrm{C}(G \times G)$ is surjective is equivalent to injectivity of the corresponding continuous map $G \times G \to G \times G$.

Remark

It follows that the density of $\Delta(\mathrm{C}(G))(\mathbb{1} \otimes \mathrm{C}(G))$ in $\mathrm{C}(G) \otimes \mathrm{C}(G)$ is equivalent to the condition that for any $p, q, p' \in G$ we have

$$\big(pq = p'q\big) \implies \big(p = p'\big)$$

and the density of $(\mathrm{C}(G) \otimes \mathbb{1})\Delta(C(G))$ in $\mathrm{C}(G) \otimes \mathrm{C}(G)$ is equivalent to

$$\big(pq = pq'\big) \implies \big(q = q'\big)$$

for any $p, q, q' \in G$.

Hence the density conditions from the definition of a compact quantum group are often referred to as the **cancellation laws**.

Theorem

Let S be a compact topological semigroup obeying cancellation laws. Then S is a compact group.

Remark

The assumption of compactness of S is crucial: the locally compact, but non-compact semigroup $\mathbb{N}$ obeys cancellation laws, but it is not a group.

Corollary

Let $\mathbb{G}$ be a compact quantum group such that the C*-algebra $\mathrm{C}(\mathbb{G})$ is commutative. Then there exists a unique compact group G such that $\mathbb{G} = G$ in the sense that $\mathrm{C}(\mathbb{G}) = \mathrm{C}(G)$ and Δ is obtained from the multiplication in G as above.

Example: dual of a discrete group

Let Γ be a discrete group and consider the left regular representation λ of Γ on the Hilbert space $\ell^2(\Gamma)$:

$$(\lambda_p\psi)(q) = \psi(p^{-1}q), \qquad \psi \in \ell^2(\Gamma),\ p, q \in \Gamma.$$

Furthermore let $C^*_r(\Gamma)$ be the C*-algebra generated by the operators $\{\lambda_p | p \in \Gamma\}$ (i.e. the smallest closed subalgebra of $B(\ell^2(\Gamma))$ containing these operators which happens to coincide with the closure of the span of $\{\lambda_p | p \in \Gamma\}$).

Next consider the unitary operator W on $\ell^2(\Gamma) \otimes \ell^2(\Gamma) \cong \ell^2(\Gamma \times \gamma)$ defined by

$$(W\Psi)(p, q) = \Psi(p, q^{-1}p), \qquad \Psi \in \ell^2(\Gamma \times \Gamma),\ p, q, \in \Gamma.$$

Then we have $(W^*\Psi)(x, y) = \Psi(x, xy^{-1})$ and consequently

$$\begin{aligned}(W(\lambda_p \otimes \mathbb{1})W^*\Psi)(a, b) &= ((\lambda_p \otimes \mathbb{1})W^*\Psi)(a, b^{-1}a) \\ &= (W^*\Psi)(p^{-1}a, b^{-1}a) = \Psi(p^{-1}a, p^{-1}b) = ((\lambda_p \otimes \lambda_p)\Psi)(a, b).\end{aligned}$$

Example: dual of a discrete group

It follows that the map $x \mapsto W(\mathbb{1} \otimes x)W^*$ restricted to $C_r^*(\Gamma)$ has range in $C_r^*(\Gamma) \otimes C_r^*(\Gamma)$ and maps each λ_p to $\lambda_p \otimes \lambda_p$. Defining Δ to be this mapping we easily see that Δ is a coassociative unital $*$-homomorphism. Moreover for any $p, q, a, b \in \Gamma$

$$\Delta(\lambda_p)(\mathbb{1} \otimes \lambda_{p^{-1}q}) = \lambda_p \otimes \lambda_q \quad \text{and} \quad (\lambda_{ab^{-1}} \otimes \mathbb{1})\Delta(\lambda_b) = \lambda_a \otimes \lambda_b$$

and such operators span a dense subspace of $C_r^*(\Gamma) \otimes C_r^*(\Gamma)$. It follows that the density conditions from the definition of a compact quantum group are satisfied, i.e. the pair $(C_r^*(\Gamma), \Delta)$ describes a compact quantum group. This compact quantum group is usually denoted by the symbol $\widehat{\Gamma}$ and it is referred to as the **dual** of Γ or, more precisely, the **reduced dual** of Γ.