Compact quantum groups - lecture 2

Faculty of Physics, University of Warsaw

Piotr M. Sołtan

Department of Mathematical Methods in Physics

Fall semester 2026/2027

Definition

A **C*-algebra** is a Banach $*$-algebra A such that

$$\forall\, a \in A \quad \|a^* a\| = \|a\|^2.$$

$\rightsquigarrow$ A **Banach algebra** is an algebra A over $\mathbb{C}$ equipped with a norm $\|\cdot\|$ such that

$$\forall\, a, b \in A \quad \|ab\| \leq \|a\|\|b\|$$

and the normed space $(A, \|\cdot\|)$ is complete.

$\rightsquigarrow$ A **Banach $*$-algebra** is a Banach algebra A with an antilinear, antimultiplicative involution $A \ni a \mapsto a^* \in A$ such that

$$\forall\, a \in A \quad \|a^*\| = \|a\|.$$

$\rightsquigarrow$ A Banach algebra is **unital** if there exists an element $\mathbb{1} \in A$ such that

$$\forall\, a \in A \quad a\mathbb{1} = \mathbb{1}a = a.$$

Examples

1. Let X be a compact space. The Banach space $\mathrm{C}(X)$ (with norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$) is a C^*-algebra with standard multiplication of functions and involution $f \mapsto f^*$ defined by

$$f^*(x) = \overline{f(x)}, \qquad x \in X.$$

   This C^*-algebra is unital and commutative.

2. Let Y be a locally compact non-compact space. Then the Banach space $\mathrm{C}_0(Y)$ of continuous functions on Y vanishing at infinity ($\forall\, \varepsilon > 0$ the set $\{y \in Y \,|\, |f(y)| \geq \varepsilon\}$ is compact) with involution $f \mapsto f^*$ as above is a non-unital commutative C^*-algebra.

Examples

3. Let $\mathscr{H}$ be a Hilbert space. Then the Banach space $\mathrm{B}(\mathscr{H})$ of all bounded operators on $\mathscr{H}$ is a Banach algebra with product given by composition of operators and the operator norm. Furthermore it is a C*-algebra with involution $\mathrm{B}(\mathscr{H}) \ni T \mapsto T^* \in \mathrm{B}(\mathscr{H})$. This C*-algebra is unital and non-commutative (unless $\dim \mathscr{H} = 1$).
4. Let $\mathscr{H}$ be an infinite-dimensional Hilbert space and let $\mathrm{K}(\mathscr{H})$ be the space of all compact operators on $\mathscr{H}$. Then $\mathrm{K}(\mathscr{H})$ is a closed $*$-subalgebra of $\mathrm{B}(\mathscr{H})$ and hence it is a C*-algebra. Moreover $\mathrm{K}(\mathscr{H})$ is not unital.
5. Any closed $*$-subalgebra of a C*-algebra is a C*-algebra. Hence in particular any closed $*$-subalgebra of $\mathrm{B}(\mathscr{H})$ is a C*-algebra.

Theorem (Gelfand-Naimark)

Let A be a C*-algebra. Then there exists a Hilbert space $\mathscr{H}$ and an isometric $*$-homomorphism $\pi\colon A \to \mathrm{B}(\mathscr{H})$ with closed range.

$\rightsquigarrow$ Consequently any C*-algebra is isometrically isomorphic to a closed $*$-subalgebra of $\mathrm{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$.

$\rightsquigarrow$ Let A be a C*-algebra. Continuous homomorphisms $A \to \mathbb{C}$ are called **characters**.

Proposition

Every character $\omega\colon A \to \mathbb{C}$ satisfies $\|\omega\| = 1$ and $\omega(a^*) = \overline{\omega(a)}$ for all $a \in A$.

Proof

If $|\omega(x)| > 1$ for some $x \in A$ with $\|x\| \leq 1$ then $|\omega(x^n)| = |\omega(x)|^n \xrightarrow[n\to\infty]{} \infty$ while $\|x^n\| \leq 1$ which contradicts boundedness of ω.

To see that $\omega(a^*) = \overline{\omega(a)}$ for all a it is enough to show that $\omega(a) \in \mathbb{R}$ for self-adjoint a.

So let us fix $a = a^*$ and let $u_t = \mathrm{e}^{\mathrm{i}ta}$ ($t \in \mathbb{R}$). Then each u_t is unitary and, in particular, $\|u_t\| \leq 1$. Hence

$$1 \geq \left|\omega(u_t)\right| = \left|\omega\Big(\sum_{n=0}^{\infty} \frac{(\mathrm{i}ta)^n}{n!}\Big)\right| = \left|\sum_{n=0}^{\infty} \frac{(\mathrm{i}t\omega(a))^n}{n!}\right| = \left|\mathrm{e}^{\mathrm{i}t\omega(a)}\right| = \mathrm{e}^{-t\,\mathrm{Im}\,\omega(a)}.$$

Since this is true for all $t \in \mathbb{R}$, we must have $\mathrm{Im}\,\omega(a) = 0$. □

- Let A be a unital C*-algebra. Then the set $\hat{A}$ of all characters of A is a subset of A_1^* – the closed unit ball of the Banach space A^*.
- Moreover, $\hat{A}$ is compact in the weak*-topology on A_1^* (the topology defined by the seminorms $A^* \ni \varphi \mapsto |\varphi(a)| \in [0, +\infty[$ $(a \in A)$.

Definition

The **Gelfand transform** of $a \in A$ is the function $\hat{a}\colon \hat{A} \to \mathbb{C}$ defined by

$$\hat{a}(\omega) = \omega(a), \qquad \omega \in \hat{A}.$$

- The Gelfand transform of any $a \in A$ is a continuous function on the compact space $\hat{A}$.
- The mapping $A \ni a \mapsto \hat{a} \in \mathrm{C}(\hat{A})$ is a unital $*$-homomorphism and $\|\hat{a}\|_\infty \leq \|a\|$ for all $a \in A$ (in fact any $*$-homomorphism between C*-algebras is automatically contractive).

Theorem (Gelfand)

Let A be a unital commutative C*-algebra Then the Gelfand transform $A \ni a \mapsto \hat{a} \in \mathrm{C}(\hat{A})$ is an isometric $*$-isomorphisms onto $\mathrm{C}(\hat{A})$.

Corollary

Let A be a unital commutative C*-algebra. Then

1. there exists a compact space X and an isometric $*$-isomorphisms $A \cong \mathrm{C}(X)$.
2. The compact space X as in 1 is unique up to homeomorphism.

- Consider two compact spaces X and Y and let $\psi\colon X \to Y$ be a continuous map. Then the mapping
$$\Psi\colon\ \mathrm{C}(Y) \ni f \longmapsto f \circ \psi \in \mathrm{C}(X)$$
is a unital $*$-homomorphism.
- Let $\Phi\colon\ \mathrm{C}(Y) \to \mathrm{C}(X)$ be a unital $*$-homomorphism. Then there exists a unique $\varphi\colon X \to Y$ such that
$$\forall\, f \in \mathrm{C}(Y) \quad \Phi(f) = f \circ \varphi.$$
Moreover Φ is injective iff φ is surjective and Φ is surjective iff φ is injective.

- Let A and B be C*-algebras and let us choose embeddings $A \subset \mathrm{B}(\mathscr{H})$ and $B \subset \mathrm{B}(\mathscr{K})$ for some Hilbert space $\mathscr{H}$ and $\mathscr{K}$.
- The **tensor product** $A \otimes B$ of A and B is defined to be the smallest C*-subalgebra of $\mathrm{B}(\mathscr{H} \otimes \mathscr{K})$ containing all operators of the form $a \otimes b$ with $a \in A$ and $b \in B$.
- The C*-algebra $A \otimes B$ is independent of the choice of embeddings $A \subset \mathrm{B}(\mathscr{H})$ and $B \subset \mathrm{B}(\mathscr{K})$.
- We denote the algebraic tensor product of A and B by $A \otimes_{\mathrm{alg}} B$. It can be shown that $A \otimes_{\mathrm{alg}} B$ is a dense $*$-subalgebra of $A \otimes B$.
- The tensor product $\mathrm{C}(X) \otimes \mathrm{C}(Y)$ is canonically isomorphic to $\mathrm{C}(X \times Y)$.
- If A is a C*-algebra then the tensor product $A \otimes \mathrm{C}(X)$ is canonically isomorphic to the algebra of all continuous functions $F\colon X \to A$ with pointwise algebraic operations and norm

$$\|F\| = \sup_{x \in X} \big\| F(x) \big\|.$$

- For any C*-algebra the tensor product $\mathrm{Mat}_n(\mathbb{C}) \otimes A$ is isomorphic to the algebra $\mathrm{Mat}_n(A)$ of $n \times n$ matrices with entries from the algebra A.
- In particular the tensor product $\mathrm{Mat}_n(\mathbb{C}) \otimes \mathrm{C}(X)$ is naturally isomorphic to the algebra of all continuous $\mathrm{Mat}_n(\mathbb{C})$-valued functions on X as well as the algebra of $n \times n$ matrices over $\mathrm{C}(X)$.