Compact quantum groups - lecture 1

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↝ Let

$$\mathsf{SU}(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, |a|^2 + |c|^2 = 1 = |b|^2 + |d|^2, \; \overline{a}b + \overline{c}d = 0, \; ad - bc = 1 \right\}$$
$$= \left\{ \begin{bmatrix} a & -\overline{c} \\ c & \overline{a} \end{bmatrix} \middle| \, |a|^2 + |c|^2 = 1 \right\}$$

↝ SU(2) is a group with respect to multiplication of matrices.

↝ Consider two functions $\alpha, \gamma \colon \mathsf{SU}(2) \to \mathbb{C}$

$$\alpha\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a, \qquad \gamma\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = c$$

and let α^* and γ^* be the corresponding complex conjugate functions:

$$\alpha^*\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \overline{a}, \qquad \gamma^*\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \overline{c}.$$

- Let $\mathrm{Pol}(\mathsf{SU}(2))$ be the set of all polynomials in α, γ, α^* and γ^*, i.e. the span of all monomials of the form $\alpha^k \gamma^l \alpha^{*m} \gamma^{*n}$ $(k, l, m, n \in \mathbb{Z}_+)$.
- We have $\alpha^* \alpha + \gamma^* \gamma = \mathbb{1}$, so for example

$$\alpha^4 \gamma^7 \alpha^{*3} \gamma^{*2} = (\alpha^* \alpha)^2 \alpha \gamma^7 \gamma^{*2} = (\mathbb{1} - \gamma^* \gamma) \alpha \gamma^7 \gamma^{*2} = \alpha \gamma^7 \gamma^{*2} - \alpha \gamma^8 \gamma^{*3}.$$

- It follows that $\mathrm{Pol}(\mathsf{SU}(2))$ is the span of all monomials of the form $\alpha^k \gamma^m \gamma^{*n}$ $(k, m, n \in \mathbb{Z}_+)$ and $\alpha^{*l} \gamma^m \gamma^{*n}$ $(l \in \mathbb{N}, m, n \in \mathbb{Z}_+)$.
- The (algebraic) tensor product $\mathrm{Pol}(\mathsf{SU}(2)) \otimes_{\mathrm{alg}} \mathrm{Pol}(\mathsf{SU}(2))$ is canonically identified with the space of polynomial functions of two variables:

$$(f \otimes g)(A, B) = f(A) g(B), \qquad f, g \in \mathrm{Pol}(\mathsf{SU}(2)),\ A, B \in \mathsf{SU}(2).$$

- The multiplication of matrices defines a mapping $\Delta\colon\ \mathrm{Pol}(\mathsf{SU}(2)) \to \mathrm{Pol}(\mathsf{SU}(2)) \otimes_{\mathrm{alg}} \mathrm{Pol}(\mathsf{SU}(2))$ by the formula

$$\Delta(f)(A, B) = f(AB).$$

⇝ To see that Δ has range in $\mathrm{Pol}(\mathsf{SU}(2)) \otimes_{\mathrm{alg}} \mathrm{Pol}(\mathsf{SU}(2))$ note that

- Δ is linear,
- Δ is multiplicative,
- $\Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}$,
- for any $f \in \mathrm{Pol}(\mathsf{SU}(2))$ we have $\Delta(f^*) = \Delta(f)^*$ (where f^* is the complex conjugate of f),
- for any $A, B \in \mathsf{SU}(2)$, $A = \left[\begin{smallmatrix} a & -\overline{c} \\ c & \overline{a} \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} x & -\overline{y} \\ y & \overline{x} \end{smallmatrix}\right]$ we have

$$\begin{aligned}\Delta(\alpha)(A,B) = \alpha(AB) = \alpha\left(\begin{bmatrix} ax - \overline{c}y & -\overline{cx} - a\overline{y} \\ cx + \overline{a}y & \overline{ax} - c\overline{y} \end{bmatrix}\right) = ax - \overline{c}y \\ = \alpha(A)\alpha(B) - \gamma^*(A)\gamma(B) = (\alpha \otimes \alpha - \gamma^* \otimes \gamma)(A,B),\end{aligned}$$

  so that $\Delta(\alpha) = \alpha \otimes \alpha - \gamma^* \otimes \gamma$,
- similarly $\Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$.

↝ Associativity of multiplication in $\mathsf{SU}(2)$ yields

$$\begin{aligned}((\Delta \otimes \mathrm{id})\Delta(f))(A,B,C) &= \Delta(f)(AB,C) = f((AB)C)\\ &= f(A(BC)) = \Delta(f)(A,BC) = ((\mathrm{id}\otimes\Delta)\Delta(f))(A,B,C),\end{aligned}$$

which means that Δ is **coassociative**:

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta.$$

↝ Δ is called the **coproduct** or **comultiplication**.

↝ The neutral element $\mathbb{1} \in \mathsf{SU}(2)$ defines a linear functional ε on $\mathrm{Pol}(\mathsf{SU}(2))$ by

$$\varepsilon(f) = f(\mathbb{1}).$$

↝ We have $\varepsilon(\alpha) = 1$, $\varepsilon(\gamma) = 0$.

↝ Clearly ε is a unital $*$-homomorphism $\mathrm{Pol}(\mathsf{SU}(2)) \to \mathbb{C}$.

⇝ For any $A \in \mathsf{SU}(2)$

$$((\varepsilon \otimes \mathrm{id})\Delta(f))(A) = \Delta(f)(\mathbb{1}, A) = f(A)$$

i.e.

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}.$$

⇝ Similarly $(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$.

Exercises

1. Let $m\colon \mathrm{Pol}(\mathsf{SU}(2)) \otimes_{\mathrm{alg}} \mathrm{Pol}(\mathsf{SU}(2)) \to \mathrm{Pol}(\mathsf{SU}(2))$ be the multiplication map $f \otimes g \mapsto fg$. Then

$$m(F)(A) = F(A, A)$$

for all $F \in \mathrm{Pol}(\mathsf{SU}(2)) \otimes_{\mathrm{alg}} \mathrm{Pol}(\mathsf{SU}(2))$ and all $A \in \mathsf{SU}(2)$.

2. For $f \in \mathrm{Pol}(SU(2))$ let $S(f)$ be the function

$$\mathsf{SU}(2) \ni A \longmapsto f(A^{-1}) \in \mathbb{C}.$$

Then $S(f) \in \mathrm{Pol}(\mathsf{SU}(2))$ and for any $f \in \mathrm{Pol}(\mathsf{SU}(2))$

$$m(S \otimes \mathrm{id})\Delta(f) = \varepsilon(f)\mathbb{1},$$
$$m(\mathrm{id} \otimes S)\Delta(f) = \varepsilon(f)\mathbb{1}.$$

$\rightsquigarrow$ The maps ε and S are called the **counit** and the **antipode** (or **coinverse**).

Definition

A **Hopf algebra** is a unital algebra $\mathscr{A}$ together with linear maps

$$\Delta\colon \mathscr{A} \longrightarrow \mathscr{A} \otimes_{\mathrm{alg}} \mathscr{A}, \quad \varepsilon\colon \mathscr{A} \longrightarrow \mathbb{C}, \quad S\colon \mathscr{A} \longrightarrow \mathscr{A}$$

such that

1. Δ is a coassociative unital homomorphism,
2. $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta$,
3. $\forall\, a \in \mathscr{A} \quad m(S \otimes \mathrm{id})\Delta(a) = \varepsilon(a)\mathbb{1} = m(\mathrm{id} \otimes S)\Delta(a)$.

Fact

Let $(\mathscr{A}, \Delta, \varepsilon, S)$ be a Hopf algebra. Then

1. ε is a unital homomorphism,
2. S is an anti-homomorphism.
3. If $(\mathscr{A}, \Delta, \varepsilon', S')$ is a Hopf algebra then $\varepsilon' = \varepsilon$ and $S' = S$.

- Due to 3 above we usually write that $(\mathscr{A}, \Delta)$ is a Hopf algebra (as ε and S are uniquely determined by $\mathscr{A}$ and Δ if they exist).
- A Hopf algebra $(\mathscr{A}, \Delta)$ is a **Hopf $*$-algebra** if $\mathscr{A}$ is a $*$-algebra and Δ is a $*$-homomorphism.
- if $(\mathscr{A}, \Delta)$ is a Hopf $*$-algebra then ε is a $*$-homomorphism and

$$\forall\, a \in \mathscr{A} \quad S(S(a^*)^*) = a.$$

- Clearly $\mathrm{Pol}(\mathsf{SU}(2))$ is a Hopf $*$-algebra (with the comultiplication Δ described earlier).

Theorem

Let $(\mathscr{A}, \Delta)$ be a Hopf algebra. Then the mappings $T_1, T_2 \colon \mathscr{A} \otimes_{\text{alg}} \mathscr{A} \to \mathscr{A} \otimes_{\text{alg}} \mathscr{A}$ determined by

$$T_1(a \otimes b) = \Delta(a)(\mathbb{1} \otimes b), \quad \text{and} \quad T_2(a \otimes b) = (a \otimes \mathbb{1})\Delta(b)$$

are bijective.

Proof

Define $R_1 \colon \mathscr{A} \otimes_{\text{alg}} \mathscr{A} \to \mathscr{A} \otimes_{\text{alg}} \mathscr{A}$ to be the unique linear map such that

$$R_1(a \otimes b) = ((\mathrm{id} \otimes S)\Delta(a))(\mathbb{1} \otimes b), \qquad a, b \in \mathscr{A}.$$

Then $R_1 \circ T_1 = \mathrm{id}_{\mathscr{A} \otimes_{\text{alg}} \mathscr{A}} = T_1 \circ R_1$, i.e. R_1 is the inverse of T_1. Similarly the inverse of T_2 is R_2 determined by

$$R_2(a \otimes b) = (a \otimes \mathbb{1})((\mathrm{id} \otimes S)\Delta(b)), \qquad a, b \in \mathscr{A}.$$

□

Remark

Let $\mathscr{A}$ be a unital algebra equipped with a coassociative homomorphism $\Delta\colon \mathscr{A} \to \mathscr{A} \otimes_{\text{alg}} \mathscr{A}$ such that the maps T_1 and T_2 described in the previous theorem are bijections $\mathscr{A} \otimes_{\text{alg}} \mathscr{A} \to \mathscr{A} \otimes_{\text{alg}} \mathscr{A}$. Then there exist maps $\varepsilon\colon \mathscr{A} \to \mathbb{C}$ and $S\colon \mathscr{A} \to \mathscr{A}$ making $(\mathscr{A}, \Delta)$ into a Hopf algebra. If additionally $\mathscr{A}$ is a $*$-algebra and Δ is a $*$-homomorphism then $(\mathscr{A}, \Delta)$ is a Hopf $*$-algebra.

- Consider the Hopf $*$-algebra $\mathrm{Pol}(\mathsf{SU}(2))$.
- For any $A, B \in \mathsf{SU}(2)$ and $f, g \in \mathrm{Pol}(\mathsf{SU}(2))$ we have

$$(T_1(f \otimes g))(A, B) = (\Delta(f)(\mathbb{1} \otimes g))(A, B) = \Delta(f)(A, B)g(B) = f(AB)g(B) = (f \otimes g)(AB, B).$$

- It follows that

$$T_1(F)(A, B) = F(AB, B)$$

for any $F \in \mathrm{Pol}(\mathsf{SU}(2)) \otimes_{\mathrm{alg}} \mathrm{Pol}(\mathsf{SU}(2))$ and $A, B \in \mathsf{SU}(2)$.

- Similarly for $F \in \mathrm{Pol}(\mathsf{SU}(2)) \otimes_{\mathrm{alg}} \mathrm{Pol}(\mathsf{SU}(2))$ and $A, B \in \mathsf{SU}(2)$ we have

$$T_2(F)(A, B) = F(A, AB).$$

- The $*$-algebra $\mathrm{Pol}(\mathsf{SU}(2))$ is generated by α and γ.
- The elements $\alpha, \gamma \in \mathrm{Pol}(\mathsf{SU}(2))$ satisfy the following relations

$$\begin{aligned} \alpha^*\alpha + \gamma^*\gamma &= \mathbb{1}, \\ \alpha\gamma = \gamma\alpha, \quad \alpha\gamma^* &= \gamma^*\alpha, \\ \alpha\alpha^* = \alpha^*\alpha, \quad \gamma\gamma^* &= \gamma^*\gamma. \end{aligned} \tag{$\star$}$$

- It can be shown that $\mathrm{Pol}(\mathsf{SU}(2))$ is the universal $*$-algebra generated by two elements α and γ satisfying $(\star)$.

$\rightsquigarrow$ Fix $q \in [-1, 1]$ and let $\mathscr{A}_q$ be the $*$-algebra generated by α and γ such that

$$\begin{gathered} \alpha^*\alpha + \gamma^*\gamma = \mathbb{1}, \quad \alpha\alpha^* + q^2\gamma\gamma^* = \mathbb{1}, \\ \alpha\gamma = q\gamma\alpha, \qquad \alpha\gamma^* = q\gamma^*\alpha, \\ \gamma^*\gamma = \gamma\gamma^*. \end{gathered} \tag{$\star_q$}$$

Exercise

The relations $(\star_q)$ are equivalent to the fact that the matrix

$$u = \begin{bmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{bmatrix} \in \mathrm{Mat}_2(\mathscr{A}_q)$$

is unitary.

$\rightsquigarrow$ It follows that for any $*$-algebra $\mathscr{B}$ and any two elements $\tilde{\alpha}, \tilde{\gamma} \in \mathscr{B}$ such that

$$\begin{bmatrix} \tilde{\alpha} & -q\tilde{\gamma}^* \\ \tilde{\gamma} & \tilde{\alpha}^* \end{bmatrix}$$

is a unitary element of $\mathrm{Mat}_2(\mathscr{B})$ then there exists a unique $*$-homomorphism

$$\Phi\colon \mathscr{A}_q \longrightarrow \mathscr{B}$$

such that $\Phi(\alpha) = \tilde{\alpha}$ and $\Phi(\gamma) = \tilde{\gamma}$.

$\rightsquigarrow$ Consider the $*$-algebra $\mathscr{B} = \mathscr{A}_q \otimes_{\text{alg}} \mathscr{A}_q$. and the matrix

$$v = \begin{bmatrix} x & -qy^* \\ y & x^* \end{bmatrix},$$

where $x = \alpha \otimes \alpha - q\gamma^* \otimes \gamma$ and $y = \gamma \otimes \alpha + \alpha^* \otimes \gamma$.

$\rightsquigarrow$ The matrix v is a unitary element of $\text{Mat}_2(\mathscr{B})$ because

$$v = \begin{bmatrix} \alpha \otimes \mathbb{1} & -q\gamma^* \otimes \mathbb{1} \\ \gamma \otimes \mathbb{1} & \alpha^* \otimes \mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} \otimes \alpha & -q\mathbb{1} \otimes \gamma^* \\ \mathbb{1} \otimes \gamma & \mathbb{1} \otimes \alpha^* \end{bmatrix}.$$

$\rightsquigarrow$ Hence there exists a unique $\Delta\colon \mathscr{A}_q \to \mathscr{A}_q \otimes_{\text{alg}} \mathscr{A}_q$ such that

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Exercises

1. The mapping $\Delta\colon \mathscr{A}_q \to \mathscr{A}_q \otimes_{\text{alg}} \mathscr{A}_q$ is coassociative.
2. There exists a unique $*$-homomorphism $\varepsilon\colon \mathscr{A} \to \mathbb{C}$ such that

$$\varepsilon(\alpha) = 1, \quad \varepsilon(\gamma) = 0$$

   This homomorphism satisfies $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta$.
3. If $q \neq 0$ there exists $S\colon \mathscr{A}_q \to \mathscr{A}_q$ such that

$$m(S \otimes \mathrm{id})\Delta(a) = \varepsilon(a)\mathbb{1} = m(\mathrm{id} \otimes S)\Delta(a)$$

   for all $a \in \mathscr{A}_q$.
4. If $q = 0$ then $(\mathscr{A}_q, \Delta)$ is not a Hopf algebra.