

Fermions

$$c_i |n_1 \dots n_i \dots\rangle = n_i (-1)^{\phi} |n_1 \dots n_i - 1 \dots\rangle$$

$$c_i^\dagger |n_1 \dots n_i \dots\rangle = (1 - n_i) (-1)^{\phi} |n_1 \dots (n_i + 1) \dots\rangle$$

$$\phi = \sum_{j < i} n_j$$

1. Calculate

$$c_5 |1111110\rangle = |1111010\rangle$$

$$c_4 |111010\dots\rangle = 0$$

$$\begin{aligned} c_2^\dagger c_3 c_1^\dagger c_2 c_3^\dagger c_1 |1100\dots\rangle &= c_2^\dagger c_3 c_1^\dagger c_2 (-|0110\dots\rangle) \\ &= c_2^\dagger c_3 (-|1010\dots\rangle) = -|1100\dots\rangle \end{aligned}$$

2. Calculate $\langle \hat{N} \rangle$ on $|\Psi\rangle = \frac{1}{\sqrt{2}} (c_3^\dagger c_2^\dagger + c_3^\dagger c_2^\dagger c_1) |0\rangle$

3. With explicit calculation for $N=2$ fermions show that $\underline{\text{I}}$ & $\underline{\text{II}}$ quantisation is equivalent.

$$\underline{\text{II}} \quad \hat{O} = \sum_{i,j} O_{i,j} c_{i,j}^\dagger c_{j,i}$$

$$\begin{aligned} \langle 1_m \dots 1_n | \hat{O} | 1_i \dots 1_j \rangle &= \langle 1_m 1_n | \sum_{i',j'} O_{i',j'} c_{i',j'}^\dagger c_{j',i'} | 1_i 1_j \rangle = \\ &= \sum_{i',j'} O_{i',j'} (\delta_{j,m} \delta_{i',n} \delta_{j',i} - \delta_{n,i} \delta_{i',m} \delta_{j',j} - \delta_{j,m} \delta_{i',n} \delta_{j',i} + \delta_{i,m} \delta_{j',j} \delta_{i',n}) \end{aligned}$$

$$\begin{aligned}
& \hat{O} |1_m \dots 1_m | 1_i \dots 1_i\rangle = \\
& = \frac{1}{2} \int d1 d2 \left(\phi_m^*(1) \phi_m^*(2) - \phi_m^*(2) \phi_m^*(1) \right) \left[O(1) + O(2) \right] \\
& \quad \left(\phi_i(1) \phi_j(2) - \phi_i(2) \phi_j(1) \right) \\
& = \frac{1}{2} \left(\underbrace{O_{mi} \delta_{mj}}_{\text{green}} - \underbrace{O_{mj} \delta_{mi}}_{\text{red}} - \underbrace{O_{ni} \delta_{mj}}_{\text{blue}} + \underbrace{O_{nj} \delta_{im}}_{\text{orange}} \right. \\
& \quad \left. + \underbrace{O_{nj} \delta_{mi}}_{\text{orange}} - \underbrace{O_{mi} \delta_{mj}}_{\text{blue}} - \underbrace{O_{mj} \delta_{ni}}_{\text{red}} + \underbrace{O_{ni} \delta_{nj}}_{\text{green}} \right) = \\
& = O_{mi} \delta_{mj} - O_{mj} \delta_{mi} - O_{ni} \delta_{mj} + O_{nj} \delta_{im}
\end{aligned}$$

Bosons

$$b_i^+ |n_1 \dots n_i \dots\rangle = \sqrt{n_i+1} |n_1 \dots (n_i+1) \dots\rangle$$

$$b_i |n_1 \dots n_i \dots\rangle = \sqrt{n_i} |n_1 \dots (n_i-1) \dots\rangle$$

4. Normalize state $|\phi\rangle = C (b_3^+ b_2^+ - b_1^+ (b_2^+)^2 b_1^+) |0\rangle$

Then calculate $\langle \hat{N} \rangle = \langle \sum_i \hat{n}_i \rangle$

Wick's theorem

Let's start by introducing contractions

$$\overbrace{\hat{A} \hat{B}} = \hat{A} \hat{B} - : \hat{A} \hat{B} :$$

Normal ordering of operators

$$\overbrace{a_i a_j} = 0$$

$$\overbrace{a_i^+ a_j^+} = 0$$

$$\overbrace{a_i^+ a_j} = 0$$

$$\overbrace{a_i a_j^+} = a_i a_j^+ - : a_i a_j^+ : = \begin{cases} a_i a_j^+ - a_j^+ a_i = \delta_{ij} & \text{Bosons} \\ a_i a_j^+ - (-a_j^+ a_i) = \delta_{ij} & \text{Fermions} \end{cases}$$

Theorem

$$A_1 \dots A_n = : A_1 \dots A_n : + \sum_{1 \text{ contraction}} \frac{+}{-} : \dots : + \sum_{2 \text{ contractions}} \frac{+}{-} : \dots : + \dots$$

For fermions: Permute pair so each contracted pair is adjacent
& multiply by $(-1)^{\# \text{ of transpositions}}$

Example

Bosons

$$\begin{aligned} a_i a_j^+ a_k a_l^+ &= : a_i a_j^+ a_k a_l^+ : + : \overbrace{a_i a_j^+} a_k a_l^+ : + : a_i a_j^+ \overbrace{a_k a_l^+} : \\ &\quad + : \overbrace{a_i a_j^+} a_k a_l^+ : + : a_i a_j^+ \overbrace{a_k a_l^+} : = \\ &= a_j^+ a_l^+ a_i a_k + \delta_{il} a_j^+ a_k + \delta_{kl} a_j^+ a_i + \delta_{ij} a_l^+ a_k + \delta_{ij} \delta_{kl} \end{aligned}$$

Fermions

$$\begin{aligned} a_i a_j^+ a_k a_l^+ &= : a_i a_j^+ a_k a_l^+ : + : \overbrace{a_i a_j^+} a_k a_l^+ : + : a_i a_j^+ \overbrace{a_k a_l^+} : \\ &\quad + : \overbrace{a_i a_j^+} a_k a_l^+ : + : a_i a_j^+ \overbrace{a_k a_l^+} : = \\ &= -a_j^+ a_l^+ a_i a_k + \delta_{il} a_j^+ a_k - \delta_{kl} a_j^+ a_i - \delta_{ij} a_l^+ a_k + \delta_{ij} \delta_{kl} \end{aligned}$$

$$c_2^\dagger c_3 c_1^\dagger c_2 c_3^\dagger c_1 = -c_2^\dagger c_1^\dagger c_3^\dagger c_3 c_2 c_1$$

$$+ \cancel{\delta_{31} : c_2^\dagger c_2 c_3^\dagger c_1 :} + \delta_{33} : c_2^\dagger c_1^\dagger c_2 c_1 : + \cancel{\delta_{23} : c_2^\dagger c_3 c_1^\dagger c_1 :} =$$

$$= c_2^\dagger c_1^\dagger c_3^\dagger c_3 c_2 c_1 + c_2^\dagger c_1^\dagger c_2 c_1$$

$$- c_2^\dagger c_1^\dagger c_1 c_2 |11\rangle = -(-1)(1)(-1) |11\rangle$$

$$\overline{c_i c_j} = c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

5. For fermions calculate

$$\begin{aligned} \text{a) } [n_i, c_j^\dagger] &= c_i^\dagger c_i c_j^\dagger - c_j^\dagger c_i^\dagger c_i = -\cancel{c_i^\dagger c_j^\dagger c_i} + c_i^\dagger \delta_{ij} - \cancel{c_j^\dagger c_i^\dagger c_i} \\ &= c_i^\dagger \delta_{ij} \end{aligned}$$

$$\text{b) } [n_i, c_j^\dagger] = -c_i \delta_{ij}$$

$$\begin{aligned} \text{c) } [n_i, n_j] &= c_i^\dagger c_i c_j^\dagger c_j - c_j^\dagger c_j c_i^\dagger c_i = \\ &= -\cancel{c_i^\dagger c_i^\dagger c_i c_j} - \cancel{c_i^\dagger c_j \delta_{ij}} + \cancel{c_j^\dagger c_i^\dagger c_j c_i} - \cancel{c_j^\dagger c_i \delta_{ji}} = 0 \end{aligned}$$