

1. Promote position and momentum to operators

$$q_j \rightarrow \hat{q}_j$$

$$[\hat{q}_i, \hat{p}_j] = i\hbar \delta_{ij}$$

$$p_j \rightarrow \hat{p}_j$$

Check if $[\hat{Q}_k, \hat{P}_{k'}]$ also fulfill canonical commutation rules

$$\begin{aligned} [Q_k, P_{k'}] &= \frac{1}{N} \left[\sum_j u_j e^{-ikR_j}, \sum_{j'} p_{j'} e^{ik'R_{j'}} \right] = \\ &= \frac{1}{N} \sum_j \sum_{j'} e^{-ikR_j} e^{ik'R_{j'}} \underbrace{[u_j, p_{j'}]}_{i\hbar \delta_{jj'}} = \\ &= \frac{1}{N} \sum_j \underbrace{e^{-i(k-k')R_j}}_{\delta_{kk'}} i\hbar = \delta_{kk'} i\hbar \end{aligned}$$

2. Using Q_k, P_k transform Hamiltonian

$$\hat{H} = \frac{1}{2m} \sum_j \hat{p}_j^2 + \frac{m\omega^2}{2} \sum_j (q_j - q_{j+1})^2$$

$$\begin{aligned} \hat{T} &= \frac{1}{2m} \sum_j \hat{p}_j^2 = \frac{1}{2mN} \sum_j \sum_{k, k'} P_k P_{k'} e^{-ikR_j} e^{ik'R_j} = \\ &= \frac{1}{2m} \sum_{k, k'} P_k P_{k'} \delta_{k, -k'} = \frac{1}{2m} \sum_k P_k P_{-k} \end{aligned}$$

$$\begin{aligned} \hat{V} &= \frac{m\omega^2}{2N} \sum_j \left(\sum_k Q_k (e^{ikR_j} - e^{ikR_{j+1}}) \right)^2 = \\ &= \frac{m\omega^2}{2N} \sum_j \sum_{k, k'} Q_k Q_{k'} \underbrace{e^{ikR_j} e^{ik'R_j}}_{N \delta_{k, -k'}} (1 - e^{ika}) (1 - e^{ik'a}) \\ &= \frac{m\omega^2}{2} \sum_k Q_k Q_{-k} (1 - \exp(-ika) - \exp(ika) + 1) \\ &= \frac{m\omega^2}{2} \sum_k Q_k Q_{-k} 2(1 - \cos(\frac{ka}{2})) = \frac{m\omega^2}{2} \sum_k Q_k Q_{-k} 4 \sin^2(\frac{ka}{2}) \end{aligned}$$

$$\mathcal{H} = \frac{1}{2m} \sum_k P_k P_{-k} + \frac{m \omega_k^2}{2} \sum_k Q_k Q_{-k} =$$

$$= \underbrace{\frac{P_0^2}{2M}}_{\text{Center of Mass}} + (k \neq 0)$$

3. Let's introduce operators

$$a_k = \sqrt{\frac{m\omega_k}{2\hbar}} \left(q_k + \frac{i}{m\omega_k} p_{-k} \right)$$

$$a_k^\dagger = \sqrt{\frac{m\omega_k}{2\hbar}} \left(q_{-k} - \frac{i}{m\omega_k} p_k \right)$$

a) Calculate $[a_k, a_{k'}^\dagger]$

$$[a_k, a_{k'}^\dagger] = \frac{m\sqrt{\omega_k \omega_{k'}}}{2\hbar} \left[q_k + \frac{i}{m\omega_k} p_{-k}; q_{-k'} - \frac{i}{m\omega_{k'}} p_{k'} \right] =$$

$$= \frac{m\sqrt{\omega_k \omega_{k'}}}{2\hbar} \left(\overset{=0}{[q_k, q_{-k'}]} + \frac{1}{m^2 \omega_k \omega_{k'}} \overset{=0}{[p_{-k}, p_{k'}]} + \frac{i}{m\omega_k} [p_{-k}, q_{-k'}] - \frac{i}{m\omega_{k'}} [q_k, p_{k'}] \right)$$

$$= \frac{m\sqrt{\omega_k \omega_{k'}}}{2\hbar} \frac{i}{\cancel{m}} \left(-\frac{i\hbar}{\omega_k} \delta_{-k, -k'} - \frac{i\hbar}{\omega_{k'}} \delta_{k, k'} \right) = \delta_{k, k'}$$

b) Use this new operators to diagonalize hamiltonian

$$\mathcal{H} = \frac{1}{2m} \sum_k P_k P_{-k} + \frac{m \omega_k^2}{2} \sum_k Q_k Q_{-k}$$

$$\begin{cases} a_k = \sqrt{\frac{m\omega_k}{2\hbar}} \left(q_k + \frac{i}{m\omega_k} p_{-k} \right) \\ a_k^\dagger = \sqrt{\frac{m\omega_k}{2\hbar}} \left(q_{-k} - \frac{i}{m\omega_k} p_k \right) \end{cases} \Rightarrow \begin{cases} q_k = \frac{1}{2} \sqrt{\frac{2\hbar}{m\omega_k}} (a_k + a_{-k}^\dagger) \\ p_k = \frac{1}{2i} \sqrt{2\hbar m \omega_k} (a_{-k} - a_k^\dagger) \end{cases}$$

$$\hat{\mathcal{H}} = \sum_k \left(-\frac{1}{\cancel{2m}} \right) \frac{1}{\cancel{4}} (\cancel{2\hbar} \cancel{m} \omega_k) (a_{-k} - a_k^\dagger)(a_k - a_{-k}^\dagger)$$

$$+ \sum_k \frac{\cancel{m} \omega_k^2}{\cancel{2}} \frac{1}{\cancel{4}} \frac{\cancel{2\hbar}}{\cancel{m} \omega_k} (a_k + a_{-k}^\dagger)(a_{-k} + a_k^\dagger) =$$

$$\begin{aligned}
&= \sum_k \left(-\frac{\hbar \omega_k}{4} \right) \left(\cancel{a_{-k}} a_k - a_{-k} a_k^\dagger - a_k^\dagger a_k + \cancel{a_k^\dagger a_{-k}} \right) \\
&+ \sum_k \left(\frac{\hbar \omega_k}{4} \right) \left(\cancel{a_k} a_{-k} + a_k a_k^\dagger + a_{-k}^\dagger a_{-k} + \cancel{a_{-k}^\dagger a_k^\dagger} \right) = \\
&= \sum_k \left(\frac{\hbar \omega_k}{4} \right) \left(a_{-k} a_{-k}^\dagger + a_k^\dagger a_k + a_k a_k^\dagger + a_{-k}^\dagger a_{-k} \right) = \\
&= \sum_k \left(\frac{\hbar \omega_k}{4} \right) \left(2 a_k a_k^\dagger + 2 a_k^\dagger a_k \right) = \\
&= \sum_k \left(\frac{\hbar \omega_k}{2} \right) \left(2 (a_k^\dagger a_k + 1) + 2 a_k^\dagger a_k \right) = \\
&= \sum_k \hbar \omega_k \left(a_k^\dagger a_k + \frac{1}{2} \right)
\end{aligned}$$