

1. Consider 1D periodic chain of masses on the spring
Find dispersion relation and speed of sound.



$$m \ddot{x}_i = -k (2x_i - x_{i-1} - x_{i+1})$$

Postulate solution in separated form:

$$x_i = R_i T(t)$$

$$m R_i \ddot{T}(t) = -k T(t) (2R_i - R_{i-1} - R_{i+1}) \quad | : T R_i m$$

$$\frac{\ddot{T}}{T} = -\frac{k}{m} \frac{1}{R_i} (2R_i - R_{i-1} - R_{i+1})$$

LHS dependent only on time

RHS dependent only on position

$$\frac{\ddot{T}}{T} = -\omega^2 = -\frac{k}{m} \frac{1}{R_i} (2R_i - R_{i-1} - R_{i+1})$$

$$T = A e^{i\omega t} + B e^{-i\omega t}$$

$$-\omega^2 = -\frac{k}{m} \frac{1}{R_i} (2R_i - R_{i-1} - R_{i+1})$$

$$m \omega^2 R_i = k (2R_i - R_{i-1} - R_{i+1})$$

We look for solution in the form

$$R_i = \lambda^i$$

$$m \omega^2 \lambda^i = k (2\lambda^i - \lambda^{i-1} - \lambda^{i+1})$$

$$m \omega^2 \lambda = k (2\lambda - 1 + \lambda^2)$$

$$\lambda^2 + (2x - mc^2)\lambda - 1 = 0$$

From Vieta formula: $x_1 x_2 = 1$

thus roots are λ_0 and λ_0^{-1} so

$$R_j = A \lambda_0^j + B \lambda_0^{-j}$$

Now use PBC $R_{j+N} = R_j$

$$A \lambda_0^{j+N} + B \lambda_0^{-j-N} = A \lambda_0^j + B \lambda_0^{-j}$$

$$A \lambda_0^j (\lambda_0^N - 1) = B \lambda_0^{-j-N} (\lambda_0^N - 1)$$

$$(\lambda_0^N - 1)(A \lambda_0^j - B \lambda_0^{-j-N}) = 0$$

$$(\lambda_0^N - 1) \lambda_0^j (A - B \lambda_0^{-2j-N}) = 0$$

$$1^\circ \lambda_0^N = 1$$

$$1 = e^{ikaN}$$

$$k = \frac{2\pi}{aN} n \quad n \in \mathbb{Z}$$

$$2^\circ \lambda_0 = 0$$

$$\Downarrow R_j = 0 \text{ or } \infty$$

$$3^\circ$$

$$B \lambda_0^{-2j-N} = A \quad \forall_j$$

$\Downarrow$ A, B are constant
 λ_0^{-2j-N} is a function of j

There is no $A \neq 0$ $B \neq 0$
 that fulfill such condition

$$\text{So finally } R_j = A e^{ika j} + B e^{-ika j}$$

$$m \omega^2 R_j = \kappa (2 R_j - R_{j+1} - R_{j-1})$$

$$m \omega^2 (A e^{i k a j} + B e^{-i k a j}) = \kappa \left[2(A e^{i k a j} + B e^{-i k a j}) - (A e^{i k a (j+1)} + B e^{-i k a (j+1)}) - (A e^{i k a (j-1)} + B e^{-i k a (j-1)}) \right]$$

$$m \omega^2 (A e^{i k a j} + B e^{-i k a j}) = \kappa \left[2(A e^{i k a j} + B e^{-i k a j}) - (A e^{i k a j} + B e^{-i k a j}) e^{i k a} - (A e^{i k a j} + B e^{-i k a j}) e^{-i k a} \right]$$

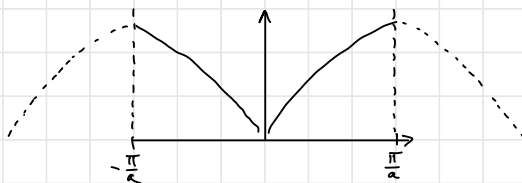
$$m \omega^2 = \kappa [2 - e^{i k a} - e^{-i k a}]$$

$$m \omega^2 = \kappa [2 - 2 \cos(ka)] \quad \left\{ \begin{array}{l} \cos 2x = 1 - 2 \sin^2 x \end{array} \right.$$

$$m \omega^2 = 2 \kappa \left[1 - 1 + 2 \sin^2\left(\frac{ka}{2}\right) \right]$$

$$m \omega^2 = 4 \kappa \sin^2\left(\frac{ka}{2}\right)$$

$$\omega = \sqrt{\frac{4 \kappa}{m}} \left| \sin\left(\frac{a k}{2}\right) \right|$$



Because for shifted $k + \frac{2\pi}{a} \equiv k$

we get the same result we constrain our consideration to $[-\frac{\pi}{a}, \frac{\pi}{a}]$

Important caveat!

For even N $n \in \{-\frac{N}{2}-1, \dots, \frac{N}{2}\}$

For odd N $n \in \{-\frac{(N-1)}{2}, \dots, \frac{(N-1)}{2}\}$

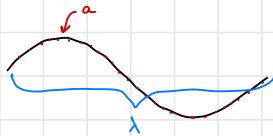
b) Calculate speed of sound

$$\text{For } \lambda \gg 2a \Rightarrow \pi \gg ka$$

$$\omega_k \stackrel{k \rightarrow 0}{\approx} \sqrt{\frac{\hbar a^2}{m}} |k| = v |k| \quad \text{Linear dispersion - massless, wave-like}$$

↑
speed of sound If dispersion is quadratic - massive, particle-like
$$\omega_k \sim \frac{\hbar^2}{2m^*} k^2$$

By sound we mean long-range oscillations



In this regime neighboring particles are not significantly different

$$\text{so: } u_{j \pm 1} \underset{\substack{\uparrow \\ \text{make it continuous}}}{\approx} u(x \pm a) \approx u(x) \pm a \partial_x a + \frac{a^2}{2} \partial_x^2 u$$

Come back to e.o.m.

$$m \partial_t^2 u = \hbar (u_{j+1} - 2u_j + u_{j-1}) \approx \hbar a^2 \partial_x^2 u$$

$$\partial_t^2 u = v^2 \partial_x^2 u \quad \left\{ \text{Wave equation} \right.$$

$$v = \sqrt{\frac{\hbar a^2}{m}}$$

In this regime dynamic is effectively wave-like

2. Discrete Fourier Transform

$$u_j = \frac{1}{\sqrt{N}} \sum_k u_k e^{ikR_j} \quad R_j = ja$$

a) Find values of k from periodic condition $u_{j+N} = u_j$

$$u_{j+N} = \frac{1}{\sqrt{N}} \sum_k u_k e^{ika(j+N)} = e^{ikaN} \frac{1}{\sqrt{N}} \sum_k u_k e^{ikaj} = u_j e^{ikaN}$$

$$e^{ikaN} = 1$$

$$k_n = \frac{2\pi}{aN} n$$

b) Find inverse transformation

$$u_j = \frac{1}{\sqrt{N}} \sum_k u_k e^{ikR_j} \quad \left| \cdot e^{-ik'R_j} \frac{1}{\sqrt{N}} \right.$$

$$\frac{1}{\sqrt{N}} e^{-ik'R_j} u_j = \frac{1}{N} \sum_k u_k e^{i(k-k')R_j} \quad \left| \sum_j \right.$$

$$\frac{1}{\sqrt{N}} \sum_j e^{-ik'R_j} u_j = \frac{1}{N} \sum_k u_k \left(\sum_j e^{i(k-k')R_j} \right) = N \delta_{kk'}$$

$$u_{k'} = \frac{1}{\sqrt{N}} \sum_j e^{-ik'R_j} u_j$$

$$\frac{1}{N} \sum_{j=0}^{N-1} e^{i(k-k')ja} = \begin{cases} 1 & k=k' \\ \frac{1 - e^{i(k-k')aN}}{1 - e^{i(k-k')a}} & k \neq k' \end{cases} = \delta_{kk'}$$

$$\left(\frac{1 - e^{i2\pi(n-m)}}{1 - e^{i\frac{2\pi}{N}(n-m)}} \right) = 0$$

c) u_j is real. What is condition on u_k to fulfill this?

$$u_j^* = u_j \Rightarrow \frac{1}{\sqrt{N}} \sum_k u_k^* e^{-ikja} = \frac{1}{\sqrt{N}} \sum_k u_k e^{ikja}$$

$$\frac{1}{\sqrt{N}} \sum_k u_{-k}^* e^{ikja} = \frac{1}{\sqrt{N}} \sum_k u_k e^{ikja}$$

$$u_{-k}^* = u_k$$