Adam Falkowski

Gravitational lensing beyond general relativity via on-shell amplitudes



Seminar at Warsaw University

12 November 2026

Plan

1. Introduction and relevant scales

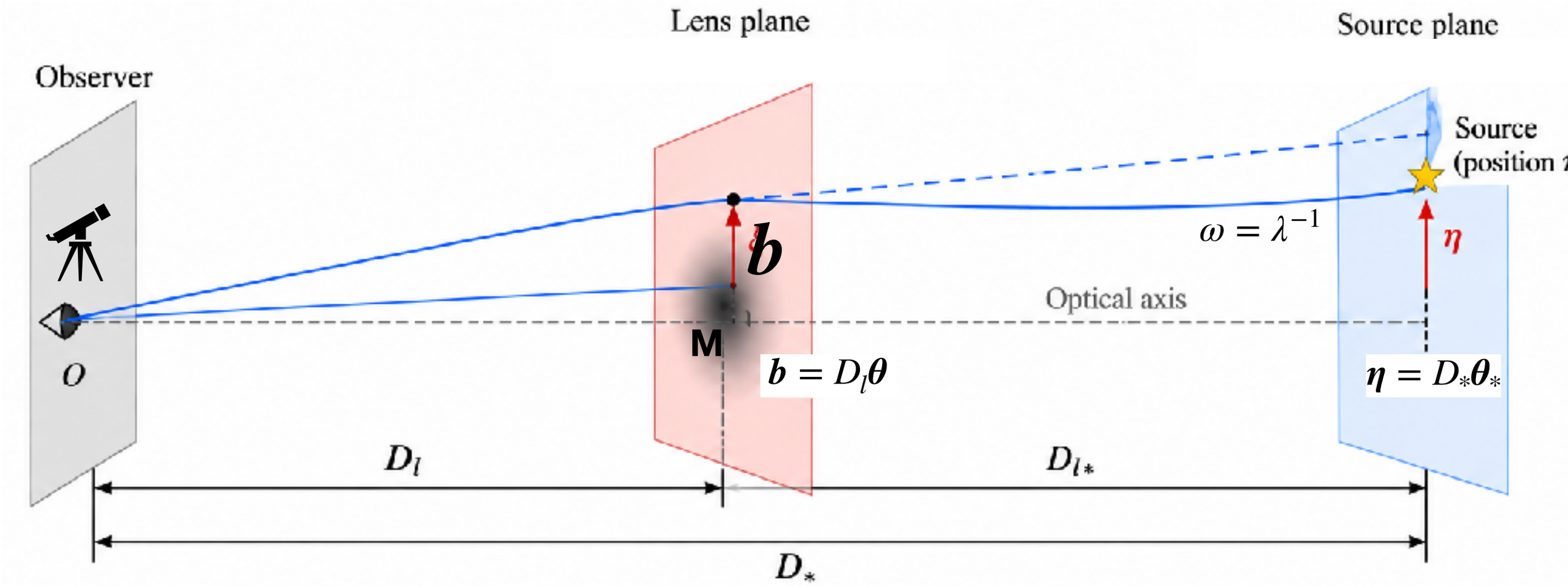
2. From on-shell amplitudes to potential and lensing

3. Long-range potential with higher-dimensional operators beyond general relativity (GR)

4. (very preliminary) applications to gravitational lensing of electromagnetic and gravitational waves

based on work with Niccolo Feci (to appear)

Settings



Ignoring cosmological expansion ($z \ll 1$)

We are interested in signal magnification by lens:

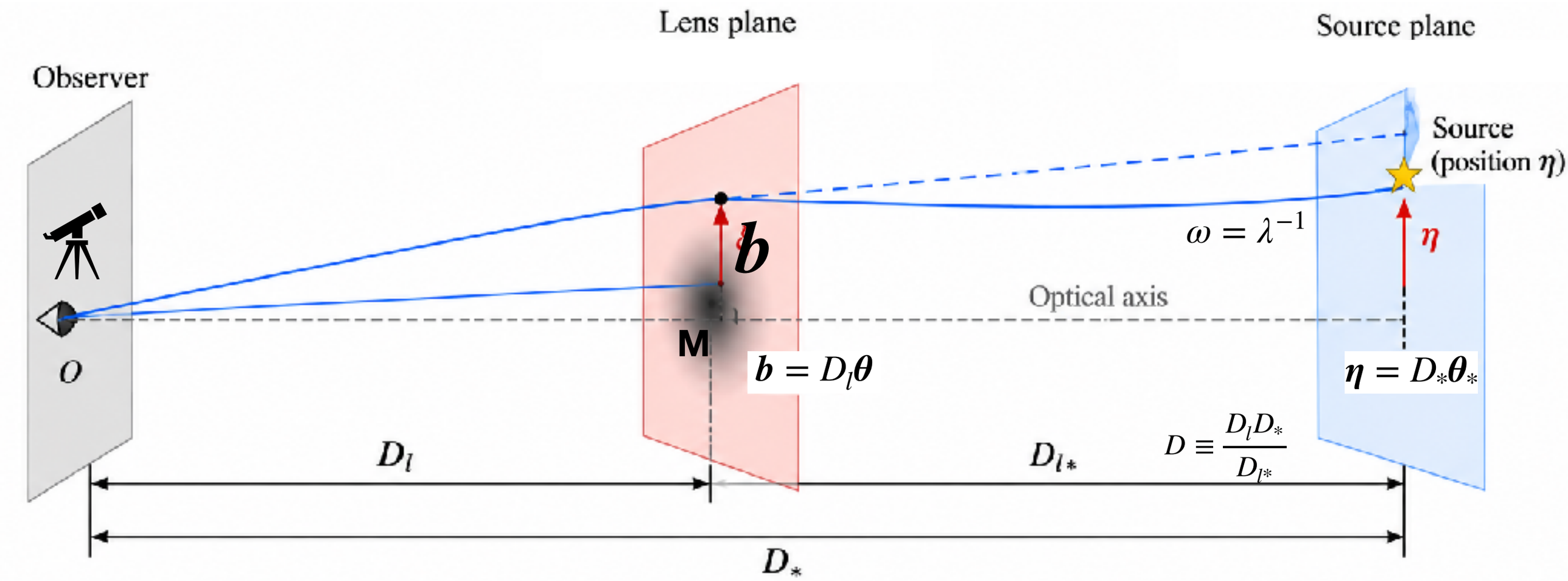
Observed wave

$$F(\boldsymbol{x}) = \phi(\boldsymbol{x})/\phi_0(\boldsymbol{x})$$

Spherical wave
from source

Nakamura, Deguchi (1999)

Standard lensing theory



In thin-lens approximation,
amplification obtained from diffraction integral

$$F(\boldsymbol{\theta}_*) = \frac{\omega D}{2\pi i}\int d^2\boldsymbol{\theta}\,\exp\left[i\omega\, t(\boldsymbol{\theta},\boldsymbol{\theta}_*)\right]$$

time of flight

In GR

$$F(\boldsymbol{\theta}_*) = \frac{\omega D}{2\pi i}\int d^2\boldsymbol{\theta}\,\exp\left[i\omega(t_{\rm geo}+t_{\rm grav})\right]$$

$$t_{\rm geo} \equiv D\frac{(\boldsymbol{\theta}-\boldsymbol{\theta}_*)^2}{2} \qquad t_{\rm grav} = -D\psi(\boldsymbol{\theta})$$

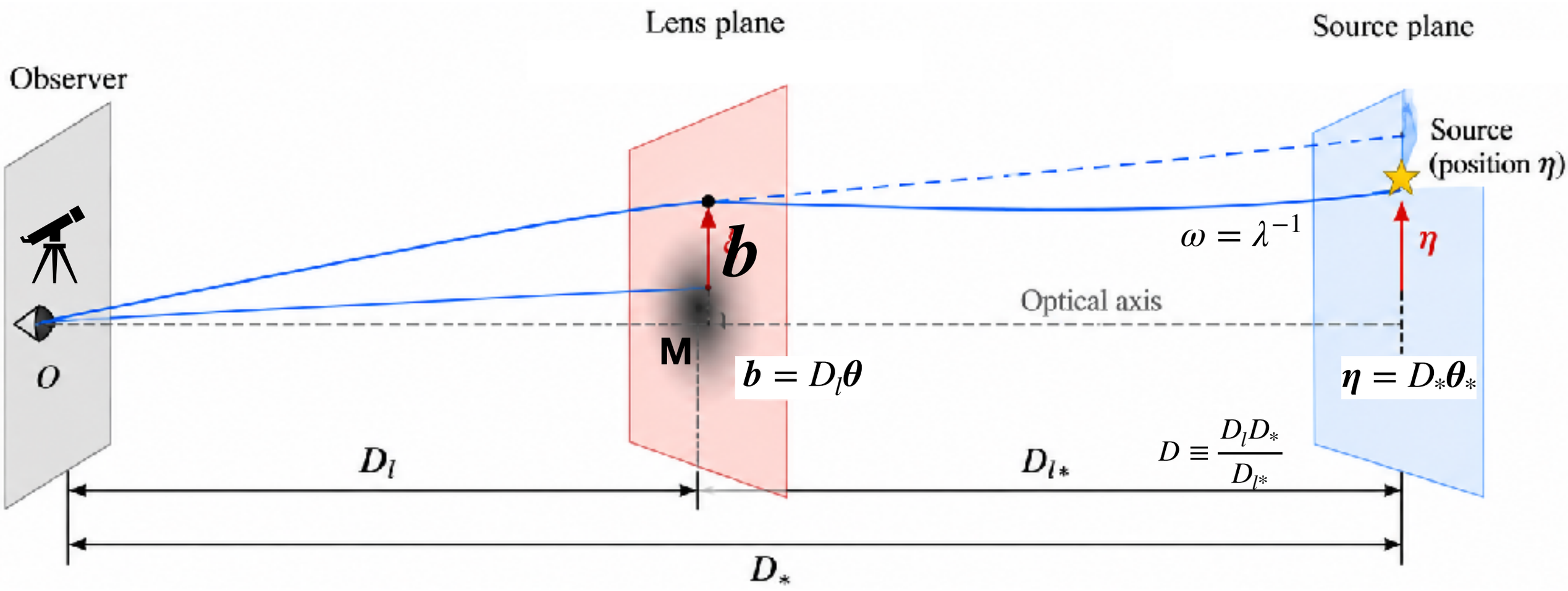
$$F(\boldsymbol{\theta}_*) = \frac{\omega D}{2\pi i}\int d^2\boldsymbol{\theta}\,\exp\left(i\omega D\left[\frac{(\boldsymbol{\theta}-\boldsymbol{\theta}_*)^2}{2} - \psi(\boldsymbol{\theta})\right]\right)$$

Fermat potential

$$\psi(\boldsymbol{\theta}) \equiv \frac{2}{D}\int_{-D_l}^{D_{l*}} dz\,\Phi(D_l\boldsymbol{\theta}, z)$$

Gravitational potential

Standard lensing theory



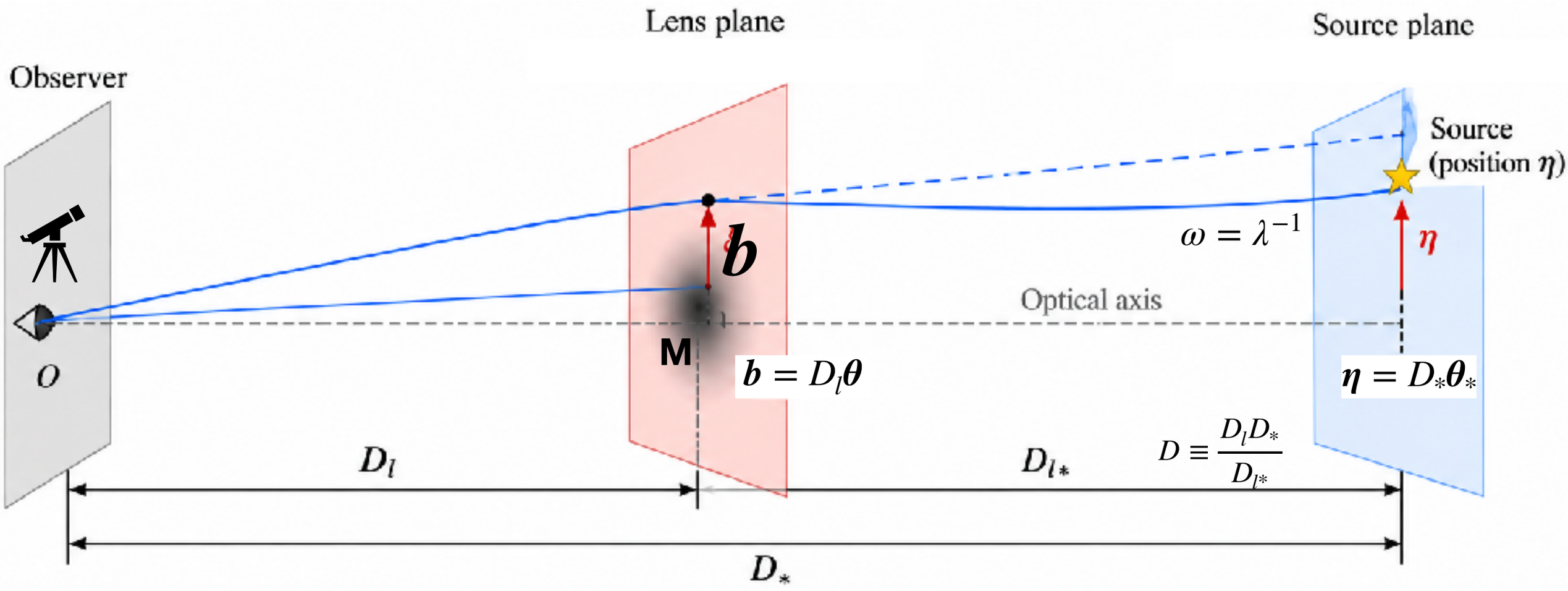
$$F(\boldsymbol{\theta}_*) = \frac{\omega D}{2\pi i}\int d^2\boldsymbol{\theta}\exp\left(i\omega D\left[\frac{(\boldsymbol{\theta}-\boldsymbol{\theta}_*)^2}{2} - \psi(\boldsymbol{\theta})\right]\right)$$

valid in both geometric optics ($\omega R_s \gg 1$) and wave optics ($\omega R_s \lesssim 1$) regimes

For geometric optics (large ω) we can use stationary phase method to derive lens equation

$$\boldsymbol{\theta} - \boldsymbol{\theta}_* = \nabla_{\boldsymbol{\theta}}\psi$$

Standard lensing theory



$$\boldsymbol{\theta} - \boldsymbol{\theta}_* = \nabla_{\boldsymbol{\theta}}\psi$$

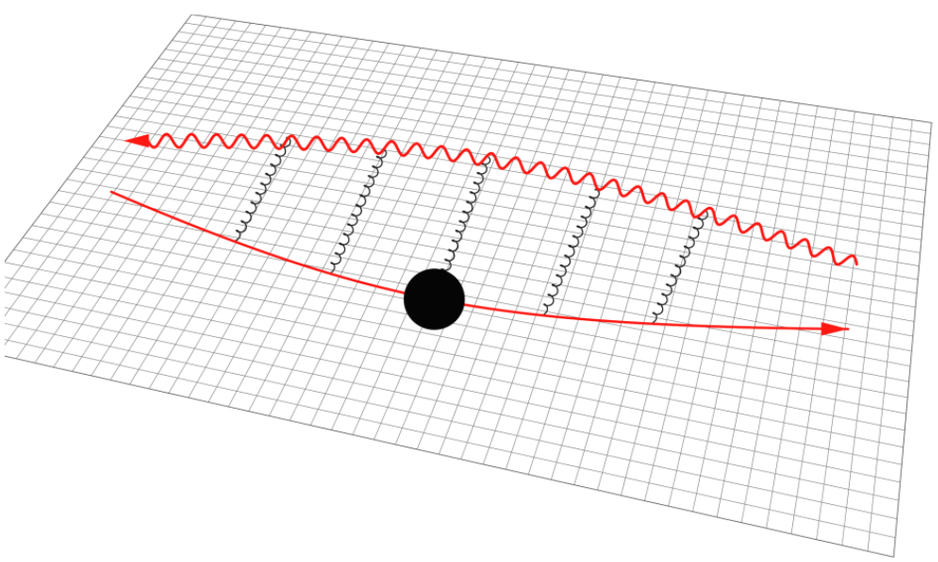
For Newtonian potential: $\Phi = -\dfrac{GM}{|\boldsymbol{x}|}$ $\quad \theta_0 \equiv 2\sqrt{D_{l*}/D_l}$

Newtonian lens equation: $\boldsymbol{\theta} - \boldsymbol{\theta}_* = \dfrac{4GM}{D}\dfrac{\boldsymbol{\theta}}{\boldsymbol{\theta}^2}$ $\quad \Rightarrow \psi(\boldsymbol{\theta}) = \dfrac{4GM}{D}\log\left(\dfrac{|\boldsymbol{\theta}|}{\theta_0}\right)$

For $\boldsymbol{\theta}_* = 0$ (perfect alignment) solution is Einstein ring: $|\boldsymbol{\theta}| = \theta_E \equiv \sqrt{\dfrac{4GM}{D}}$

For non-alignment, multiple images (2 in Newtonian case) with different magnification and time delay

Amplitudes vs Classical Gravity



Flat space scattering

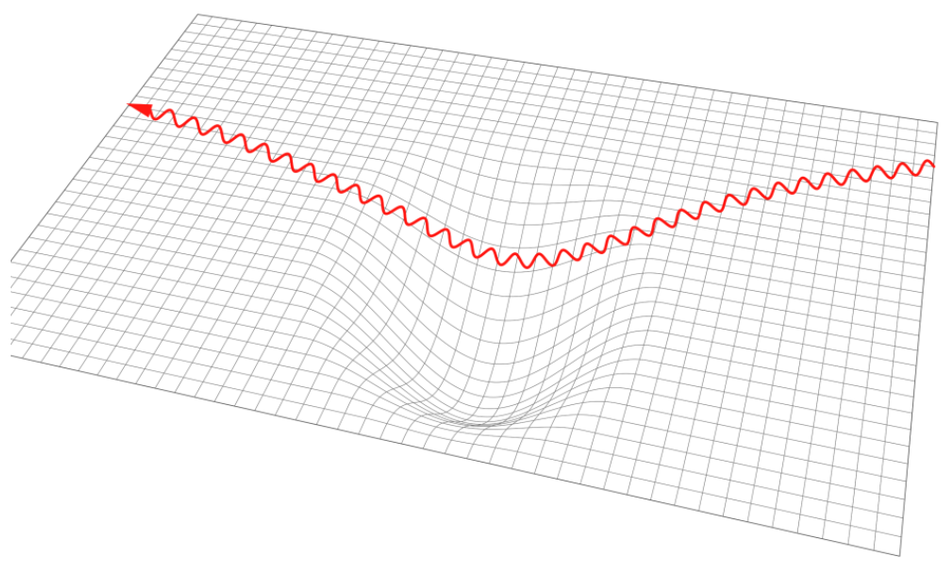


QFT amplitude

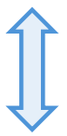
LS equation



Classical limit



Wave deflection by a potential

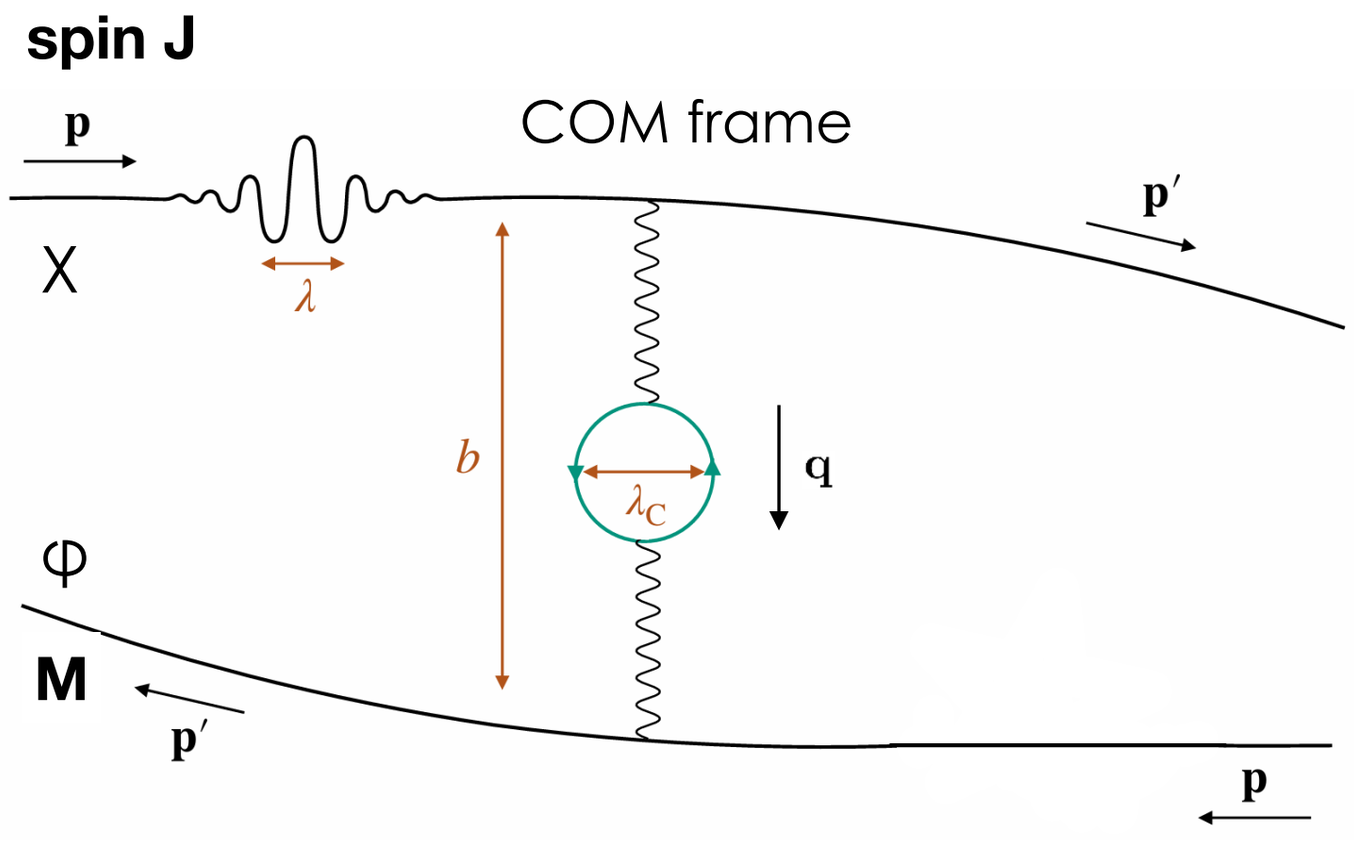


Solution of EOM

Why?

Amplitude techniques are already established for classical GR calculations
They allow one to calculate higher orders in velocity (post-Newtonian, PN) and in Newton constant G (post-Minkowskian, PM) expansion more systematically
They also greatly simplify treatment of spin

Scattering framework



$$c = \hbar = 1$$

Relevant scales

$$\lambda_C \equiv \frac{1}{M}$$

$$b \sim \frac{1}{|q|}$$

$$\lambda \equiv \frac{1}{\omega}$$

$$\cos\theta = 1 - \frac{q^2}{2\omega^2}$$

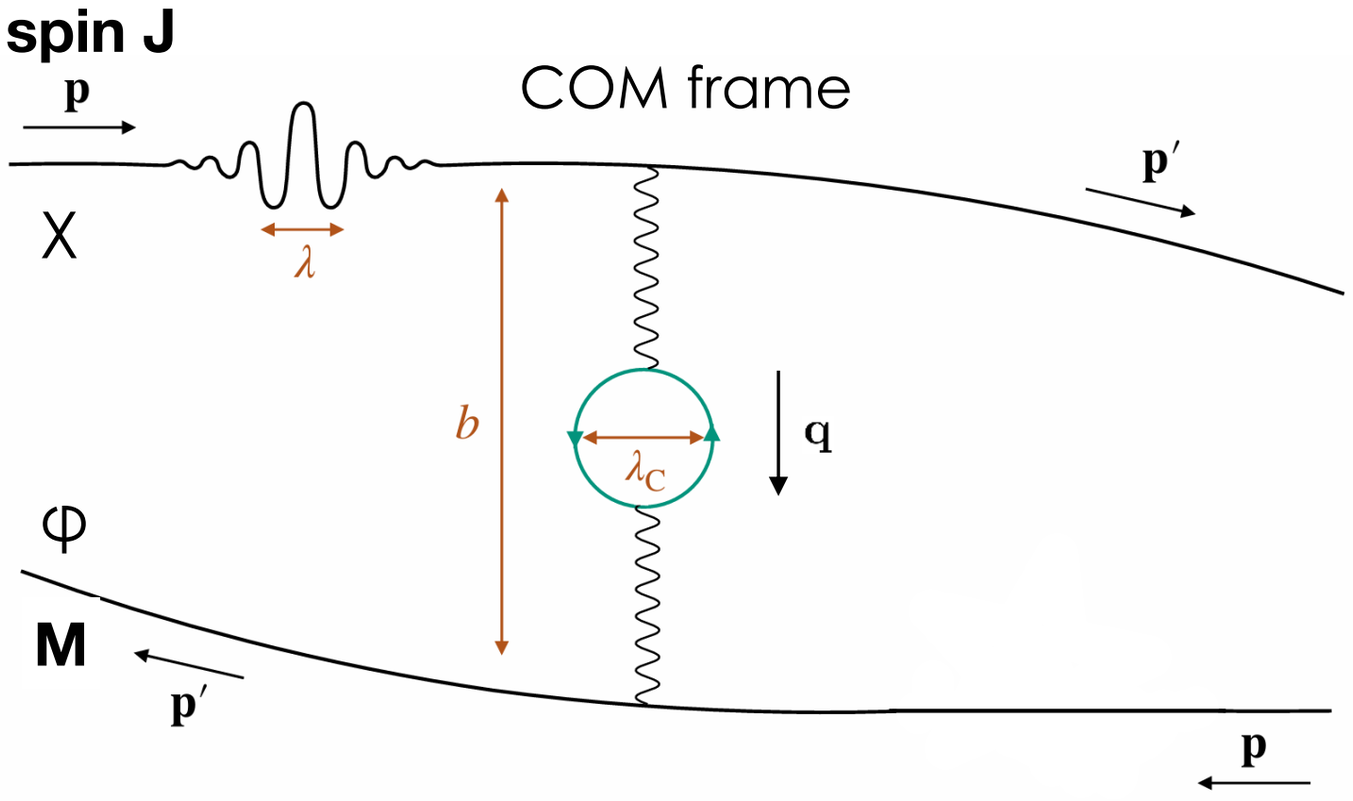
$$R_s \equiv \frac{M}{4\pi M_{\rm Pl}^2}$$

Amplitude: $\mathscr{M}[1_X^h 2_\Phi \to (-3)_X^{h'}(-4)_\Phi] = s_{h'}\mathscr{M}[1_X^h 2_\Phi 3_X^{-h'} 4_\Phi] \equiv \mathscr{M}_{h'}^h$

$$p_1 = (\omega, \boldsymbol{p}), \quad p_2 = \left(\sqrt{M^2+\omega^2}, -\boldsymbol{p}\right)$$

$$p_3 = -(\omega, \boldsymbol{p}'), \quad p_4 = -\left(\sqrt{M^2+\omega^2}, -\boldsymbol{p}'\right)$$

Scattering framework



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Classical Born regime

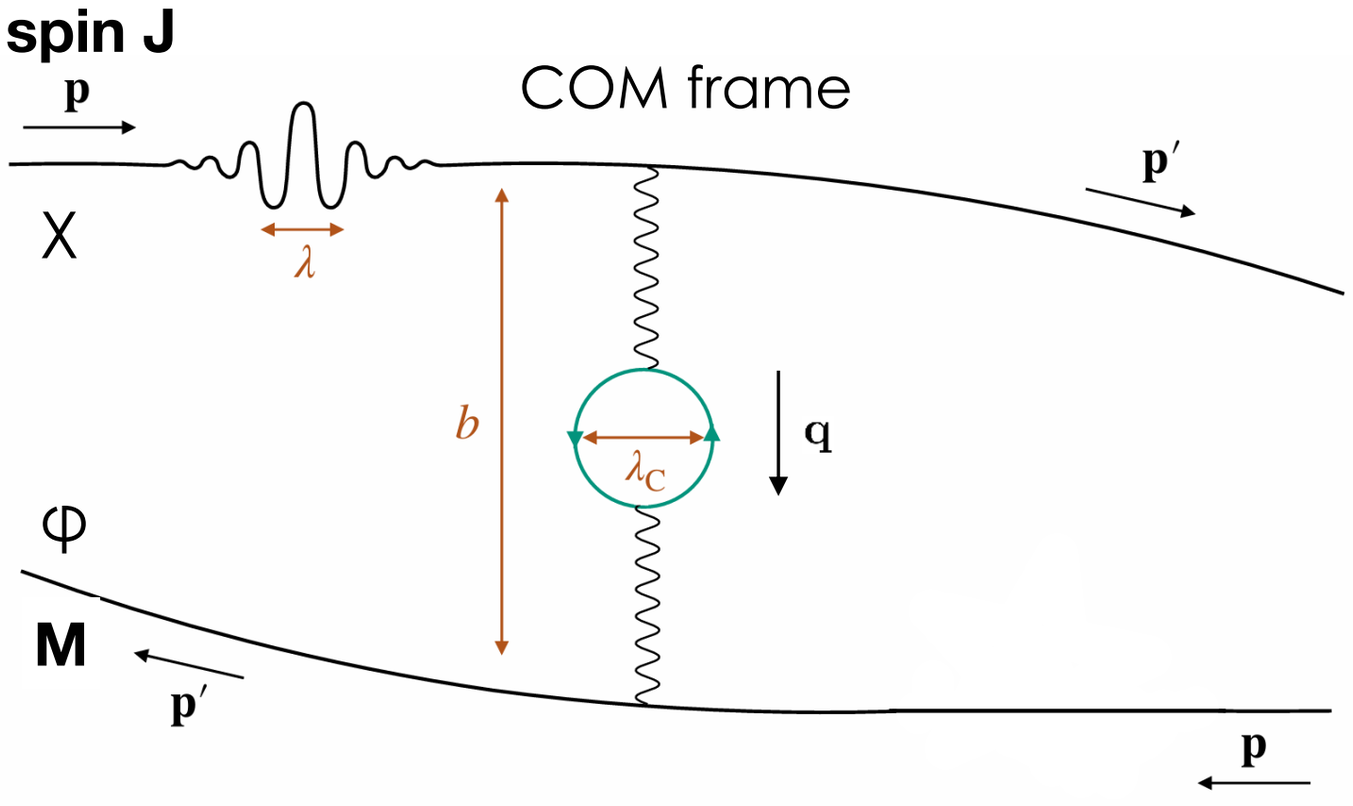
Correia, Isabella
[arXiv:2406.13737]

$$M \gg M_{\rm Pl} \qquad \omega \ll M_{\rm Pl} \qquad |\boldsymbol{b}| \gg R_s \qquad \Rightarrow \omega \ll M \quad \lambda_C \ll R_s$$

Eikonal limit: $b \gg \lambda$ $\Rightarrow |\boldsymbol{q}| \ll \omega \quad \theta \ll 1$

I will calculate amplitudes in classical Born regime, without necessarily taking eikonal limit

Scattering framework



Relevant scales

$$\lambda_C \equiv \frac{1}{M}$$

$$b \sim \frac{1}{|\boldsymbol{q}|}$$

$$\lambda \equiv \frac{1}{\omega}$$

$$\cos\theta = 1 - \frac{\boldsymbol{q}^2}{2\omega^2}$$

$$R_s \equiv \frac{M}{4\pi M_{\rm Pl}^2}$$

Classical Born regime

$$M \gg M_{\rm Pl} \qquad \omega \ll M_{\rm Pl} \qquad |\boldsymbol{b}| \gg R_s$$

Subleading corrections of order $\frac{\omega}{M}$ and $\frac{|\boldsymbol{q}|}{M}$ are quantum and will be dropped henceforth

Bellazzini et al
[arXiv:2211.00085]

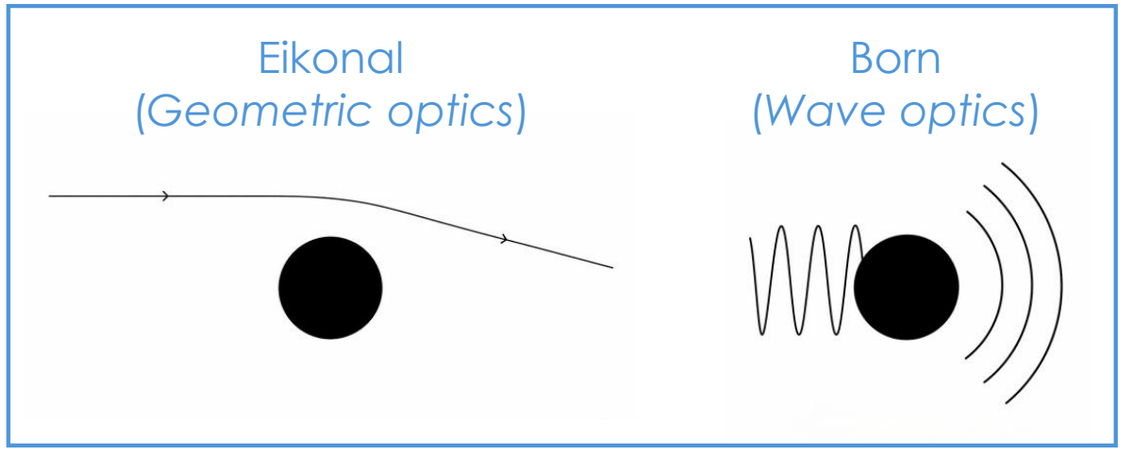
Why the Born limit?

- Less explored compared to the eikonal regime, more complicated classical limit

- Interesting feautures : breaking of gravitational coupling universality (GCU), helicity mixing effects.

⇒ Wave optics regime of GW lensing

[C. Gonzalez, Parra-Martinez, et al., 2026]
[Nakamura, Deguchi, 1999]



From on-shell amplitudes to potential and lensing

Lippmann-Schwinger equation

Correia, Isabella
[arXiv:2406.13737]
following Todorov
Phys. Rev. D 3 (1971) 2351

Given amplitude in Born regime, potential in momentum space can be defined via Lippmann-Schwinger equation

$$\mathscr{M}^{h}_{h'}(\boldsymbol{p},\boldsymbol{p}') = V^{h}_{h'}(\boldsymbol{p},\boldsymbol{p}') + \int \frac{d^d\boldsymbol{l}}{(2\pi)^d} G(\boldsymbol{p},\boldsymbol{l})\mathscr{M}^{h}_{a}(\boldsymbol{p},\boldsymbol{l})V^{a}_{h'}(\boldsymbol{l},\boldsymbol{p}')$$

Hermiticity

$$V^{h}_{h'}(\boldsymbol{p},\boldsymbol{p}')^* = V^{h'}_{h}(\boldsymbol{p}',\boldsymbol{p})$$

Regularization

$$d = 3 - 2\epsilon$$

By unitarity, $\mathrm{Im}G(\boldsymbol{p},\boldsymbol{l}) \simeq \dfrac{\pi\delta(\boldsymbol{l}^2 - \boldsymbol{p}^2)}{2M}$ **. A convenient choice of Green's function is**

$$G(\boldsymbol{p},\boldsymbol{l}) \simeq \frac{1}{2M(\boldsymbol{l}^2 - \boldsymbol{p}^2 - i0)}$$

Loop expansion:

$$\mathscr{M}^{h}_{h'} = \mathscr{M}^{(0)h}_{h'} + \mathscr{M}^{(1)h}_{h'} + \dots$$

$$V^{h}_{h'} = V^{(0)h}_{h'} + V^{(1)h}_{h'} + \dots$$

In momentum space, loop expansion in $\dfrac{M|\boldsymbol{q}|}{M_{\mathrm{Pl}}^2}$

Matching:

$$V^{(n)h}_{h'}(\boldsymbol{p},\boldsymbol{p}') = \mathscr{M}^{(n)h}_{h'} - \mathscr{I}^{(n)h}_{h'}$$

Iteration term:

$$\mathscr{I}^{(0)h}_{h'} = 0 \qquad \mathscr{I}^{(1)h}_{h'} = \frac{1}{2M}\int \frac{d^d\boldsymbol{l}}{(2\pi)^d} \frac{V^{(0)h}_{a}(\boldsymbol{p},\boldsymbol{l})V^{(0)a}_{h'}(\boldsymbol{l},\boldsymbol{p}')}{\boldsymbol{l}^2 - \boldsymbol{p}^2}$$

Wave function

Potential in position space: $$U(\boldsymbol{x}) \simeq -\frac{1}{2M}\int \frac{d^d\boldsymbol{q}}{(2\pi)^d} e^{i\boldsymbol{q}\boldsymbol{x}} V(\boldsymbol{p},\boldsymbol{p}')$$

$$U = 4\omega^2\Phi$$

In position space, loop expansion in $$\frac{M}{|\boldsymbol{x}|M_{\rm Pl}^2} = \frac{R_s}{|\boldsymbol{x}|}$$

Green's function in position space: $$G(\boldsymbol{x},\boldsymbol{y}) \simeq -2M\int \frac{d^d\boldsymbol{l}}{(2\pi)^d} e^{i\boldsymbol{l}(\boldsymbol{x}-\boldsymbol{y})} G(\boldsymbol{p},\boldsymbol{l}) \overset{d\to 3}{\to} -\frac{e^{i\omega|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}$$

Define effective one-body wave function $\Psi(\boldsymbol{x})$: $$\mathscr{M}(\boldsymbol{p},\boldsymbol{p}') \simeq -2M\int d^d\boldsymbol{y} e^{-i\boldsymbol{p}'\boldsymbol{y}} U(\boldsymbol{y})\Psi(\boldsymbol{y})$$

Wave function satisfies Schrodinger-like equation:

$$(\nabla^2 + \omega^2)\Psi(\boldsymbol{x}) = U(\boldsymbol{x})\Psi(\boldsymbol{x})$$

Wave function

$$(\nabla^2 + \omega^2)\Psi(\boldsymbol{x}) = U(\boldsymbol{x})\Psi(\boldsymbol{x})$$

Solution: $$\Psi(\boldsymbol{x}) = \Psi_0(\boldsymbol{x}) + \int d^3\boldsymbol{y} G(\boldsymbol{x},\boldsymbol{y})U(\boldsymbol{y})\Psi(\boldsymbol{y}) \qquad G(\boldsymbol{x},\boldsymbol{y}) \simeq -\frac{e^{i\omega|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}$$

Ψ_0 is solution satisfying $(\nabla^2 + \omega^2)\Psi_0(\boldsymbol{x}) = 0$, e.g. plane wave. For large $|\boldsymbol{x}|$:

$$\Psi(\boldsymbol{x}) \approx \Psi_0(\boldsymbol{x}) - \frac{e^{i\omega|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|}\int d^3\boldsymbol{y} e^{-i\boldsymbol{p}'\boldsymbol{y}}U(\boldsymbol{y})\Psi(\boldsymbol{y})$$

$$= \Psi_0(\boldsymbol{x}) + f(\boldsymbol{p},\boldsymbol{p}')\frac{e^{i\omega|\boldsymbol{x}|}}{|\boldsymbol{x}|}$$

Identify coefficient of scattered wave: $$f(\boldsymbol{p},\boldsymbol{p}') = -\frac{1}{4\pi}\int d^3\boldsymbol{y} e^{-i\boldsymbol{p}'\boldsymbol{y}}U(\boldsymbol{y})\Psi(\boldsymbol{y})$$

Compare with: $$\mathscr{M}(\boldsymbol{p},\boldsymbol{p}') \simeq -2M\int d^3\boldsymbol{y} e^{-i\boldsymbol{p}'\boldsymbol{y}}U(\boldsymbol{y})\Psi(\boldsymbol{y})$$

Therefore: $$f(\boldsymbol{p},\boldsymbol{p}') \simeq \frac{\mathscr{M}(\boldsymbol{p},\boldsymbol{p}')}{8\pi M}$$

Amplitude to lensing

Assume scalar waves for time being, $\Psi \to \phi$

Carrillo Gonzalez et al. [arXiv:2511.15797]

Partial wave expansion of plane wave:

$$\phi_{0\mathrm{pw}}(\boldsymbol{x}) = e^{i\omega z} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(\omega\,|\boldsymbol{x}\,|) P_l(\cos\hat{\theta})$$

$\hat{\theta}$ is polar angle in lens-centric frame

Partial wave expansion of spherical wave:

$$\phi_{0\mathrm{sw}}(\boldsymbol{x}) = \frac{e^{i\omega|\boldsymbol{x}-\boldsymbol{x}_*|}}{|\boldsymbol{x}-\boldsymbol{x}_*|} = \frac{i}{2}\omega \sum_{l=0}^{\infty} (-)^l (2l+1) \Big[h_l^{(1)}(\omega\,|\boldsymbol{x}\,|) + h_l^{(2)}(\omega\,|\boldsymbol{x}\,|) \Big] h_l^{(1)}(\omega D_{l*}) P_l(\cos\hat{\theta})$$

$j_l(\omega\,|\boldsymbol{x}\,|)$

Interaction add phase shift χ_l to outgoing component:

$$\phi(\boldsymbol{x}) = \frac{i}{2}\omega \sum_{l=0}^{\infty} (-)^l (2l+1) \Big[e^{i\chi_l} h_l^{(1)}(\omega\,|\boldsymbol{x}\,|) + h_l^{(2)}(\omega\,|\boldsymbol{x}\,|) \Big] h_l^{(1)}(\omega D_{l*}) P_l(\cos\hat{\theta})$$

Define scattered wave: $\phi(\boldsymbol{x}) - \phi_{0\mathrm{sw}}(\boldsymbol{x}) = f(\boldsymbol{x}) \dfrac{e^{i\omega|\boldsymbol{x}|}}{|\boldsymbol{x}\,|}$

After expanding Hankels: $f = \sum_{l=0}^{\infty} \dfrac{(2l+1)}{2i\omega} (e^{i\chi_l} - 1) \exp\left(\dfrac{i(l+1/2)^2}{2\omega} \left[\dfrac{1}{|\boldsymbol{x}\,|} + \dfrac{1}{D_{l*}} \right] \right) P_l(\cos\hat{\theta})$

Amplitude to lensing

Carrillo Gonzalez et al. [arXiv:2511.15797]

$$f = \sum_{l=0}^{\infty} \frac{(2l+1)}{2i\omega}(e^{i\chi_l} - 1)\exp\left(\frac{i(l+1/2)^2}{2\omega}\left[\frac{1}{|\boldsymbol{x}|} + \frac{1}{D_{l*}}\right]\right) P_l(\cos\hat{\theta})$$

Identify $b = (l+1/2)/\omega$, **phase shift** χ_l **becomes** $\chi^{\mathrm{eik}}(b)$

Take large l **(eikonal) limit and use** $\sum_{l=0}^{\infty}(2l+1) \to 2\omega^2 \int_0^{\infty} bdb$, $P_l(\cos\hat{\theta}) \to J_0(b\omega\hat{\theta})$

$$f^{\mathrm{eik}} = -i\omega \int_0^{\infty} bdbJ_0(b\omega\hat{\theta})\left(e^{i\chi^{\mathrm{eik}}} - 1\right)\exp\left(\frac{ib^2\omega}{2}\frac{D_*}{D_l D_{l*}}\right) \simeq -\frac{i\omega}{2\pi}\int d^2\boldsymbol{b}e^{-i\tilde{\boldsymbol{q}}\boldsymbol{b}}\left(e^{i\chi^{\mathrm{eik}}} - 1\right)\exp\left(\frac{i\boldsymbol{b}^2\omega}{2}\frac{D_*}{D_l D_{l*}}\right)$$

Rescale to amplitude $f \simeq \dfrac{\mathscr{M}(\boldsymbol{p},\boldsymbol{p}')}{8\pi M}$ $\qquad \tilde{\boldsymbol{q}} \equiv \omega\dfrac{D_*}{D_{l*}}\boldsymbol{\theta}_*$

$$\mathscr{M}^{\mathrm{eik}}(\boldsymbol{p},\boldsymbol{p}') = -4iM\omega\int d^2\boldsymbol{b}e^{-i\tilde{\boldsymbol{q}}\boldsymbol{b}}\left(e^{i\chi^{\mathrm{eik}}} - 1\right)\exp\left(\frac{i\boldsymbol{b}^2\omega}{2}\frac{D_*}{D_l D_{l*}}\right)$$

Amplitude to lensing

Carrillo Gonzalez et al. [arXiv:2511.15797]

$$\mathscr{M}^{\rm eik}(\boldsymbol{p},\boldsymbol{p}') = -4iM\omega\int d^2\boldsymbol{b}\left(e^{i\chi^{\rm eik}}-1\right)\exp\left(\frac{i\boldsymbol{b}^2\omega}{2}\frac{D_*}{D_lD_{l*}}-i\tilde{\boldsymbol{q}}\boldsymbol{b}\right)$$

$$\boldsymbol{\theta}=\frac{\boldsymbol{b}}{D_l}$$

$$\boldsymbol{\theta}_*=\frac{\boldsymbol{b}_o}{D}$$

$$\tilde{\boldsymbol{q}}\equiv\frac{\omega\boldsymbol{b}_o}{D_l}$$

Rearranging

$$\mathscr{M}^{\rm eik}(\boldsymbol{p},\boldsymbol{p}') = -4iM\omega D_l^2 e^{-\frac{i\omega D\boldsymbol{\theta}_*^2}{2}}\int d^2\boldsymbol{\theta}\left(e^{i\chi^{\rm eik}}-1\right)\exp\left(i\frac{D\omega}{2}\left(\boldsymbol{\theta}-\boldsymbol{\theta}_*\right)^2\right)$$

Compare with diffraction integral

$$F(\boldsymbol{\theta}_*)=\frac{\omega D}{2\pi i}\int d^2\boldsymbol{\theta}\exp\left(i\omega D\left[\frac{(\boldsymbol{\theta}-\boldsymbol{\theta}_*)^2}{2}-\psi(\boldsymbol{\theta})\right]\right)$$

We can identify how diffraction integral is related to amplitudes (in eikonal limit)

$$F(\boldsymbol{\theta}_*)-1=\frac{D_*}{8\pi MD_lD_{l*}}e^{\frac{i\omega D\boldsymbol{\theta}_*{}^2}{2}}\mathscr{M}^{\rm eik}(\boldsymbol{p},\boldsymbol{p}')$$

provided we identify gravitational time delay with eikonal phase of scattering amplitude

$$\psi(\boldsymbol{\theta})=-\frac{\chi^{\rm eik}}{\omega D}$$

Eikonal phase to potential

For normal amplitudes (with incoming plane waves)

$$\mathscr{M}^{\rm eik}(\boldsymbol{p},\boldsymbol{p}') = -4iM\omega\int d^2\boldsymbol{b}e^{-i\tilde{\boldsymbol{q}}\boldsymbol{b}}\left(e^{i\chi^{\rm eik}}-1\right)$$

At leading order

$$\mathscr{M}^{(0)}(\boldsymbol{p},\boldsymbol{p}') = V^{(0)}(\boldsymbol{p},\boldsymbol{p}') = 4M\omega\int d^2\boldsymbol{b}e^{-i\tilde{\boldsymbol{q}}\boldsymbol{b}}\chi^{\rm eik}$$

Going to position space

$$-\frac{1}{2\omega}\int d^3\boldsymbol{x}e^{-i\boldsymbol{q}\boldsymbol{x}}U^{(0)}(\boldsymbol{x}) = \int d^2\boldsymbol{b}e^{-i\tilde{\boldsymbol{q}}\boldsymbol{b}}\chi^{(0)}$$

Parametrizing $\boldsymbol{x} = (\boldsymbol{b}, z)$ and approximating $\boldsymbol{q}\boldsymbol{x} \approx \tilde{\boldsymbol{q}}\boldsymbol{b}$ we identify

$$\chi^{(0)} = -\frac{1}{2\omega}\int dz U^{(0)}(\boldsymbol{b}, z)$$

Finally,

$$\psi^{(0)} = \frac{1}{2\omega^2 D}\int dz U^{(0)}(\boldsymbol{b}, z)$$

This in fact generalizes to higher PM orders

Story so far

- One can systematically derive diffraction integral describing lensing from on-shell amplitudes in flat space

$$F(\boldsymbol{\theta}_*) = \frac{\omega D}{2\pi i}\int d^2\boldsymbol{\theta}\exp\left(i\omega D\left[\frac{(\boldsymbol{\theta}-\boldsymbol{\theta}_*)^2}{2}-\psi(\boldsymbol{\theta})\right]\right)$$

- Gravitational time delay in diffraction integral is related to integral of position-space potential

$$\psi = \frac{1}{2\omega^2 D}\int dz U(\boldsymbol{b},z) = -\frac{1}{4M\omega^2 D}\int\frac{d^2\tilde{\boldsymbol{q}}}{(2\pi)^2}e^{i\tilde{\boldsymbol{q}}\boldsymbol{b}}V(\boldsymbol{p},\boldsymbol{p}')\,|_{q_z=0}$$

- Momentum-space potential can be calculated from amplitudes using Lippmann-Schwinger equation by calculating amplitude for massless particle scattering off a massive one, and subtracting iteration terms:

$$V^{(n)}(\boldsymbol{p},\boldsymbol{p}') = \mathscr{M}^{(n)} - \mathscr{I}^{(n)}$$

$$\mathscr{I}^{(0)} = 0 \quad \mathscr{I}^{(1)} = \frac{1}{2M}\int\frac{d^d\boldsymbol{l}}{(2\pi)^d}\frac{V^{(0)}(\boldsymbol{p},\boldsymbol{l})V^{(0)}(\boldsymbol{l},\boldsymbol{p}')}{\boldsymbol{l}^2-\boldsymbol{p}^2}$$

It can be then Fourier-transformed to position space if needed

Tree-level potential

Gravity in on-shell formalism

For massless spin-J particle X, its minimal coupling to gravity is described by

$$\mathscr{M}[1_g^- 2_X^- 3_X^+] = -\frac{\langle 12\rangle^{2+2J}\langle 13\rangle^{2-2J}}{M_{\rm Pl}\langle 23\rangle^2} \qquad \mathscr{M}[1_g^+ 2_X^+ 3_X^-] = -\frac{[12]^{2+2J}[13]^{2-2J}}{M_{\rm Pl}[23]^2}$$

Gravity in on-shell formalism

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Here massless momentum is decomposed into 2-component spinors: $p_i^\mu \sigma_\mu = \lambda_i \tilde{\lambda}_i$

Shorthand notation: $\lambda_i = |i\rangle, \quad \tilde{\lambda}_i = |i]$ $\qquad \sigma^\mu \equiv (1,\boldsymbol{\sigma})$

Lorentz-invariant spinor contractions: $\langle ij\rangle = (\lambda_i\lambda_j) \equiv \lambda_i^2\lambda_j^1 - \lambda_i^1\lambda_j^2$

$[ij] = (\tilde{\lambda}_i\tilde{\lambda}_j) \equiv \tilde{\lambda}_i^1\tilde{\lambda}_j^2 - \tilde{\lambda}_i^2\tilde{\lambda}_j^1$

Momentum contractions: $(2p_ip_j) = \mathrm{Tr}[p_i\sigma p_j\bar{\sigma}] = \langle ij\rangle[ij]$

Spinors transform under little group: $\lambda_i \to e^{-i\alpha_i}\lambda_i \quad \tilde{\lambda}_i \to e^{i\alpha_i}\tilde{\lambda}_i$

Amplitudes also transform under little group: $\mathscr{M}[1_g^{h_1} 2_X^{h_2} 3_X^{h_3}] \to e^{4\alpha_1+2h_2\alpha_2+2h_3\alpha_3}\mathscr{M}[1_g^{h_1} 2_X^{h_2} 3_X^{h_3}]$

For 3-point amplitudes, little group scaling (almost) fixes them up to normalization

Gravity in on-shell formalism

For massless spin-$J \in [0,2]$ particle X, minimal coupling to gravity is described by

$$\mathscr{M}[1_g^- 2_X^- 3_X^+] = -\frac{\langle 12\rangle^{2+2J}\langle 13\rangle^{2-2J}}{M_{\rm Pl}\langle 23\rangle^2} \qquad \mathscr{M}[1_g^+ 2_X^+ 3_X^-] = -\frac{[12]^{2+2J}[13]^{2-2J}}{M_{\rm Pl}[23]^2}$$

For massless spin-J particle X, non-minimal coupling to gravity is described by

$$\mathscr{M}[1_g^- 2_X^- 3_X^-] = -\frac{C_X}{M_{\rm Pl}}\langle 12\rangle^2\langle 13\rangle^2\langle 23\rangle^{2J-2} \qquad \mathscr{M}[1_g^+ 2_X^+ 3_X^+] = -\frac{C_X^*}{M_{\rm Pl}}[12]^2[13]^2[23]^{2J-2}$$

This is moot for J=0 (just rescaling of minimal coupling)
It corresponds to non-local operator for J=1/2
It corresponds to local operator for J=3/2,
which however clashes with supersymmetry required by massless gravitino

For J=1,2 and real C_X this corresponds to local operators:

$$\mathscr{L} \supset \sqrt{-g}R_{\mu\nu\alpha\beta}\left(\frac{C_\gamma}{2}F^{\mu\nu}F^{\alpha\beta} - \frac{C_g M_{\rm Pl}^2}{6}R^{\alpha\beta\rho\sigma}R_{\rho\sigma}{}^{\mu\nu}\right)$$

For massive spin-zero particle, minimal coupling to gravity is described by

$$\mathscr{M}[1_g^- 2_\Phi 3_\Phi] = -\frac{\langle 1|p_2|\zeta]^2}{M_{\rm Pl}[1\zeta]^2} \qquad \mathscr{M}[1_g^+ 2_\Phi 3_\Phi] = -\frac{[1|p_2|\zeta\rangle^2}{M_{\rm Pl}\langle 1\zeta\rangle^2}$$

Beyond GR

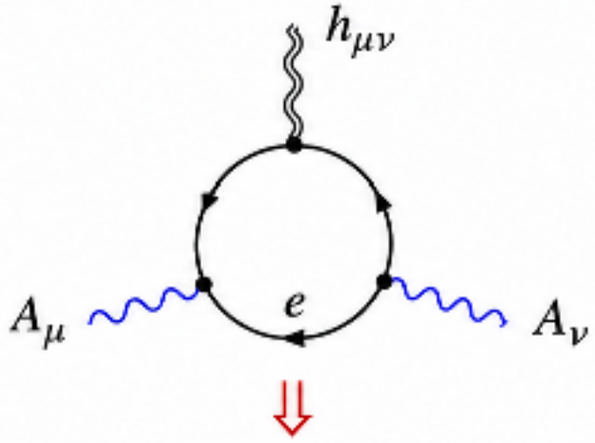
$$\mathscr{L} \supset \sqrt{-g} R_{\mu\nu\alpha\beta}\left(\frac{C_\gamma}{2} F^{\mu\nu}F^{\alpha\beta} - \frac{C_g M_{\rm Pl}^2}{6} R^{\alpha\beta\rho\sigma} R_{\rho\sigma}{}^{\mu\nu}\right)$$

This is less exotic than you think

Integrating out charged and neutral leptons → $\mathcal{L} \supset \sqrt{-g}\, R_{\mu\nu\alpha\beta}\left(\frac{C_\gamma}{2} F^{\mu\nu}F^{\alpha\beta} - \frac{C_g M_{\rm Pl}^2}{6} R^{\alpha\beta\rho\sigma} R_{\rho\sigma}{}^{\mu\nu}\right)$

Electron loop (charged lepton)

Integrate out e with $m_e \gg \omega, |q|$:

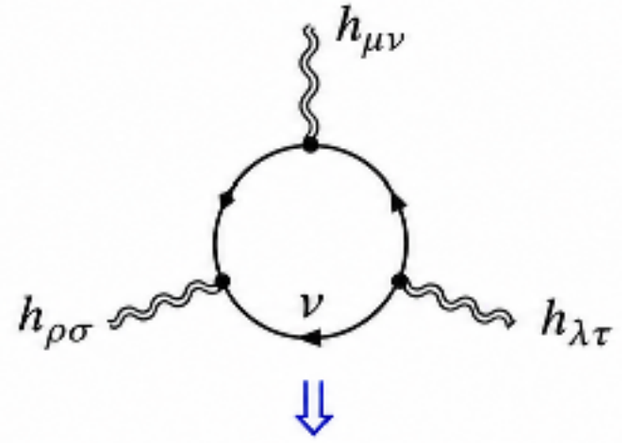


$$\mathcal{L}_{\rm eff} \supset \sqrt{-g}\,\frac{C_\gamma}{2}\, R_{\mu\nu\alpha\beta} F^{\mu\nu}F^{\alpha\beta},$$

$C_\gamma = \frac{\alpha}{45\pi m_e^2}$ (m_e : electron mass)

Neutrino loop (neutral lepton)

Integrate out ν with $m_\nu \gg \omega, |q|$:



$$\mathcal{L}_{\rm eff} \supset -\sqrt{-g}\,\frac{C_g M_{\rm Pl}^2}{6}\, R_{\mu\nu\alpha\beta} R^{\alpha\beta\rho\sigma} R_{\rho\sigma}{}^{\mu\nu},$$

$C_g = -\frac{1}{16\pi^2 M_{\rm pl}^2 m_\nu^2}\frac{1}{1260}$ (m_ν : neutrino mass)

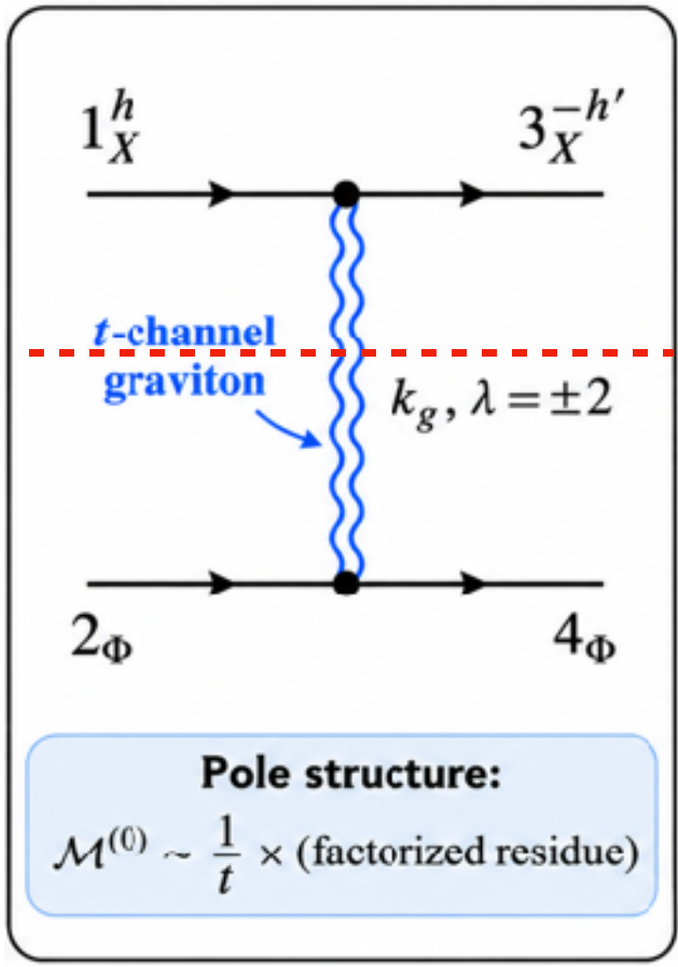
More generally, $C_g = \frac{(-)^{2S}}{16\pi^2 M_{\rm pl}^2 m^2}\frac{(2S+1)}{2520}$

Wilson coefficient of R^3 can be easily suppressed by very small (even macroscopic) scale, depending on lightest neutrino mass

Tree-level gravitational scattering

Unitarity implies factorization

$$\mathrm{Res}_{t\to 0}\mathscr{M}^{(0)}[1_X^h 2_\Phi 3_X^{-h'} 4_\Phi] = -\,\mathscr{M}[(-t)_g^{-a} 1_X^h 3_X^{h'}]\mathscr{M}[t_g^a 2_\Phi 4_\Phi]$$



We reconstruct amplitudes near t pole as

$$\mathscr{M}^{(0)}[1_X^- 2_\Phi \to (-3)_X^- (-4)_\Phi] = -\frac{\langle 1\,|\,p_2\,|\,3]^{2J}(s-M^2)^{2-2J}}{M_{\rm Pl}^2 t} + \dots$$

$$\mathscr{M}^{(0)}[1_X^+ 2_\Phi \to (-3)_X^+ (-4)_\Phi] = -\frac{\langle 3\,|\,p_2\,|\,1]^{2J}(u-M^2)^{2-2J}}{M_{\rm Pl}^2 t} + \dots$$

$$\mathscr{M}^{(0)}[1_X^- 2_\Phi \to (-3)_X^+ (-4)_\Phi] = -\frac{\langle 13\rangle^{2J}}{M_{\rm Pl}^2 t}\left[\frac{\delta_{J2}}{(s-M^2)^2} + C_X (s-M^2)^2\right] + \dots$$

$$\mathscr{M}^{(0)}[1_X^+ 2_\Phi \to (-3)_X^- (-4)_\Phi] = -\frac{[13]^{2J}}{M_{\rm Pl}^2 t}\left[\frac{\delta_{J2}}{(s-M^2)^2} + C_X^* (s-M^2)^2\right] + \dots,$$

Dots stand for contact terms regular at $t \to 0$. They do not contribute to long-range potential

Tree-level potential

Inserting kinematics and going to Born regime

$$s \simeq M^2 \quad s - M^2 \simeq 2M\omega \quad t = -\boldsymbol{q}^2$$

$$\langle 13\rangle = \frac{\boldsymbol{q}_-}{\sqrt{1-\frac{\boldsymbol{q}^2}{4\omega^2}}} \qquad [13] = \frac{\boldsymbol{q}_+}{\sqrt{1-\frac{\boldsymbol{q}^2}{4\omega^2}}} \qquad \boldsymbol{q}_\pm \boldsymbol{p} = 0 \quad \boldsymbol{q}_\pm^* = \boldsymbol{q}_\mp$$

$$\langle 1\,|p_2|\,3] \simeq 2M\omega\sqrt{1-\frac{\boldsymbol{q}^2}{4\omega^2}} \qquad [1\,|p_2|\,3\rangle \simeq -2M\omega\sqrt{1-\frac{\boldsymbol{q}^2}{4\omega^2}}$$

$$V^{(0)}(\boldsymbol{p},\boldsymbol{p}') = \frac{4M^2\omega^2}{M_{\rm Pl}^2\boldsymbol{q}^2}\begin{pmatrix} 1 & \boldsymbol{q}_-^{2J}\left[\frac{\delta_{J2}}{16\omega^4} + C_X\right] \\ \boldsymbol{q}_+^{2J}\left[\frac{\delta_{J2}}{16\omega^4} + C_X^*\right] & 1 \end{pmatrix}$$

Helicity conserving potential universal for all spins

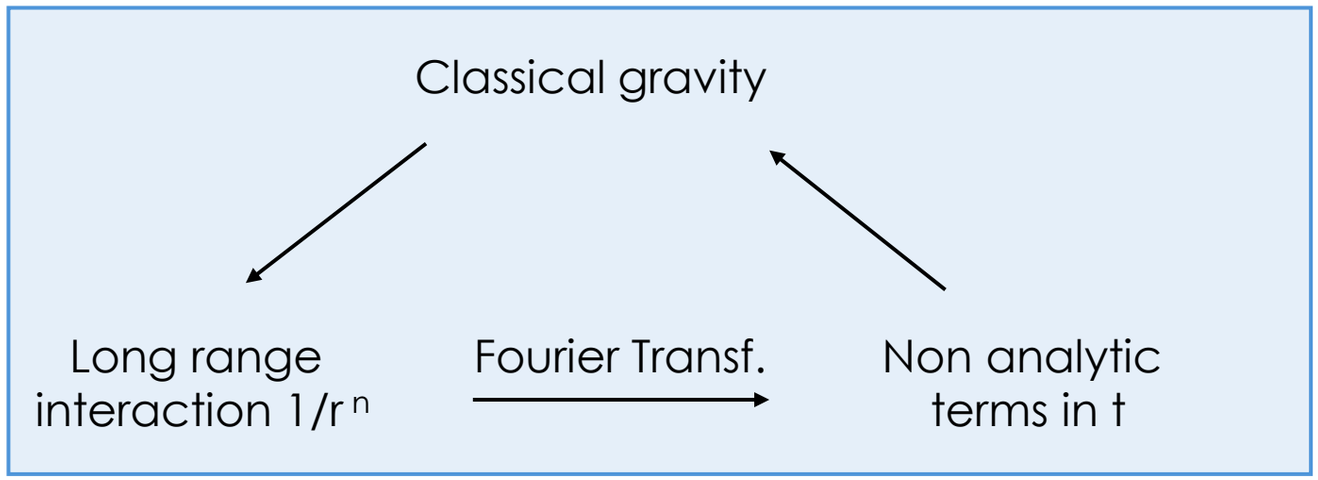
Beyond GR, helicity breaking potential appears for J=1

Beyond GR, helicity breaking potential is modified for J=2

One-loop potential

One-loop amplitudes

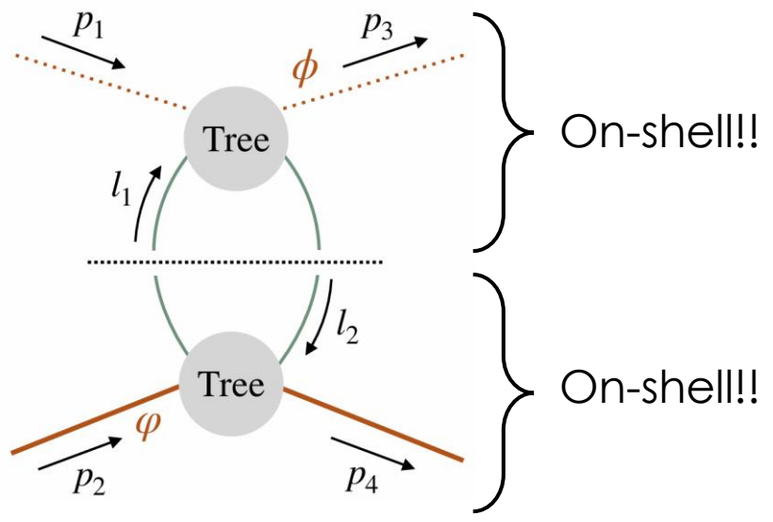
Long range interactions and on-shell amplitudes



$$t = (p_1 + p_3)^2 = -|\mathbf{p}' - \mathbf{p}|^2 = -|\mathbf{q}|^2$$

⇒ Problem fully described by the t-channel discontinuity!
[Bjerrum-Bohr, Donoghue, Vanhove, 2014]

⇒ *We can use the Spinor-helicity formalism!!*

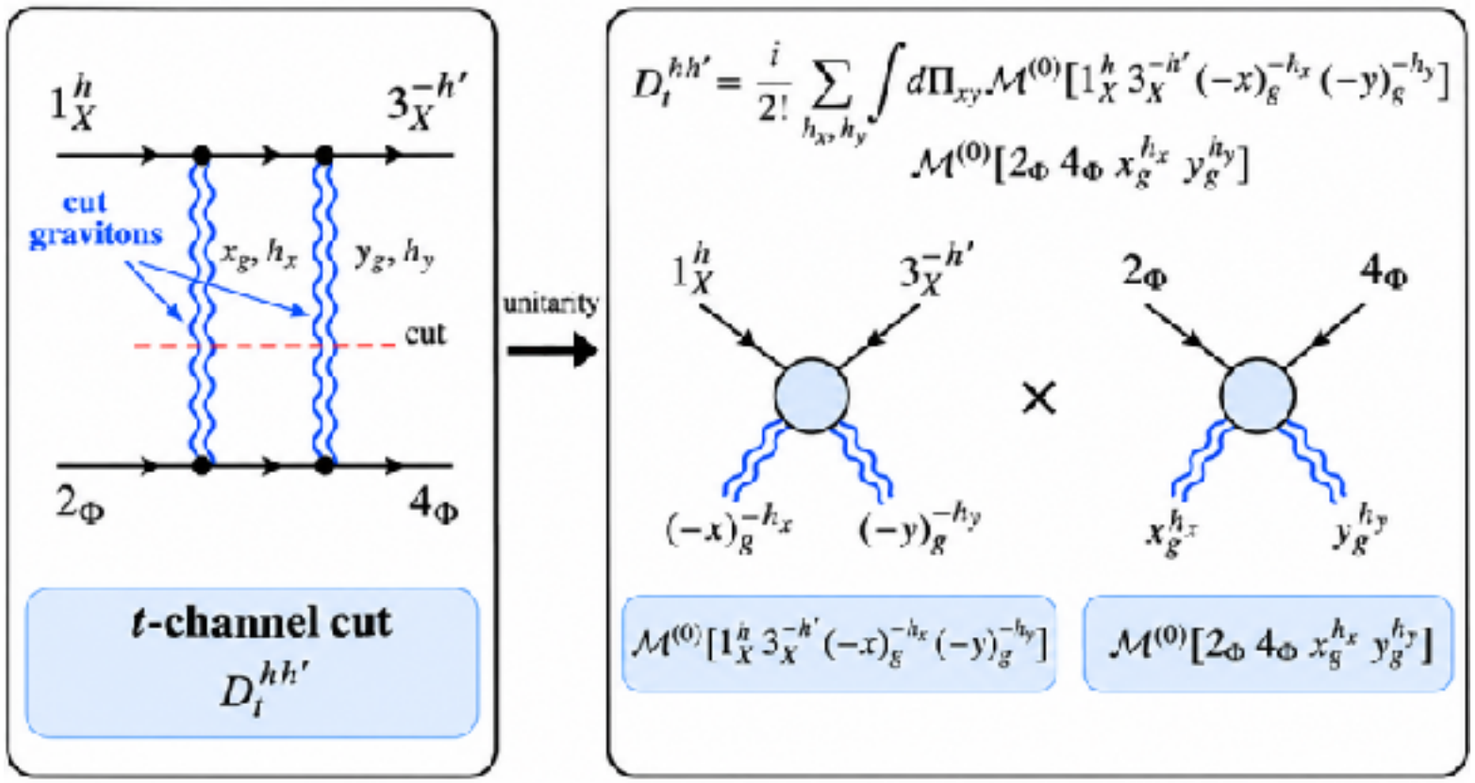


One-loop amplitudes

t-channel discontinuity of 1-loop amplitude

$$D_t^{hh'} \equiv \text{Disc}_t \mathscr{M}^{(1)}[1_X^h 2_\Phi 3_X^{-h'} 4_\Phi] \equiv \mathscr{M}^{(1)}[1_X^h 2_\Phi 3_X^{-h'} 4_\Phi](t+i\epsilon) - \mathscr{M}^{(1)}[1_X^h 2_\Phi 3_X^{-h'} 4_\Phi](t-i\epsilon)$$

Unitarity implies: $$D_t^{hh'} = \frac{i}{2!}\sum_{h_x,h_y}\int d\Pi_{xy}\mathscr{M}^{(0)}[1_X^h 3_X^{-h'}(-x)_g^{-h_x}(-y)_g^{-h_y}]\mathscr{M}^{(0)}[2_\Phi 4_\Phi x_g^{h_x} y_g^{h_y}]$$



Gravi-Compton amplitudes for cut integral:

$$\mathscr{M}^{(0)}[1_\gamma^- 2_\gamma^- 3_g^- 4_g^-] = \frac{C_\gamma}{M_{\text{Pl}}^2}\frac{\langle 12\rangle^2\langle 34\rangle^4}{s} + \ldots$$

$$\mathscr{M}^{(0)}[1_g^- 2_g^- 3_g^- 4_g^-] = -\frac{10C_g}{M_{\text{Pl}}^2 s}\langle 12\rangle^2\langle 34\rangle^2\langle 13\rangle\langle 24\rangle\langle 14\rangle\langle 23\rangle$$

$$\mathscr{M}^{(0)}[1_X^- 2_X^+ 3_g^- 4_g^+] = \frac{\langle 13\rangle^{2J}[24]^{2J}\langle 3|p_1|4]^{4-2J}}{M_{\text{Pl}}^2 stu} + \ldots$$

$$\mathscr{M}^{(0)}[1_X^- 2_X^- 3_g^- 4_g^+] = C_X\frac{\langle 12\rangle^{2J-2}\langle 13\rangle^2\langle 23\rangle^2[4|p_1p_2|4]^2}{M_{\text{Pl}}^2 stu} + \ldots$$

$$\mathscr{M}^{(0)}[1_X^- 2_X^+ 3_g^- 4_g^-] = C_g\frac{(-)^J t^{1-J}u^{1-J}\langle 34\rangle^4\langle 1|p_3|2]^{2J}}{M_{\text{Pl}}^2 s} + \ldots,$$

One-loop amplitudes

On-shell t-channel discontinuity

$$p_{nx} \equiv 2p_n p_x$$

2-body phase space

$$D_t^{--} = \frac{i(-)^{2J}}{2M_{\rm Pl}^4 t^2}\int \frac{d\Pi_{xy}}{p_{1x}p_{2x}p_{3x}p_{4x}}\left\{ \langle 1|p_x p_2 p_y|3]^{2J}\mathrm{Tr}[p_1\sigma p_y\bar{\sigma}p_2\sigma p_x\bar{\sigma}]^{4-2J} + (x\leftrightarrow y) + 2\mathrm{Re}C_g M^4 t^4 (-)^J p_{1x}^{2-J}p_{3x}^{2-J}\langle 1|p_x|3]^{2J}\right\}$$

$$D_t^{-+} = \frac{i\langle 13\rangle^{2J}}{2M_{\rm Pl}^4 t^2}\int \frac{d\Pi_{xy}}{p_{1x}p_{2x}p_{3x}p_{4x}}\left\{ \delta_{J2}M^4t^4 + C_X\left[\left(\mathrm{Tr}[p_x\sigma p_2\bar{\sigma}p_y\sigma p_1\bar{\sigma}]\right)^4 + (x\leftrightarrow y) + (9J-8)M^4t^{6-2J}p_{1x}^J p_{3x}^J\right]\right\}$$

Very easy and compact result with spinor helicity formalism

⇩

Much easier computation than using Feynmann diagrams!!!

⇒ Next step: discontinuity in terms of discontinuities of scalar master integrals.

$$D_t^{hh'} = \mathrm{Disc}_t\left\{ C_{\square}^{hh'}I_{\square}^{st} + C_{x\square}^{hh'}I_{\square}^{tu} + C_{\triangleleft}^{hh'}I_{\triangleleft}^{t} + C_{\triangleright}^{hh'}I_{\triangleright}^{t} + C_{\circ}^{hh'}I_{\circ}^{t}\right\}$$

Scalar Master integrals

$$D_t^{hh'} = \mathrm{Disc}_t\left\{ C_{\square}^{hh'} I_{\square}^{st} + C_{x\square}^{hh'} I_{\square}^{tu} + C_{\triangleleft}^{hh'} I_{\triangleleft}^{t} + C_{\triangleright}^{hh'} I_{\triangleright}^{t} + C_{\circ}^{hh'} I_{\circ}^{t} \right\}$$

$$I_{\square}^{st} \equiv \int \frac{d^d k}{i(2\pi)^d} \frac{1}{k^2 (k+p_1)^2 (k+p_1+p_3)^2 [(k-p_2)^2 - M^2]}$$

Boxes

$$I_{\square}^{tu} \equiv \int \frac{d^d k}{i(2\pi)^d} \frac{1}{k^2 (k+p_1)^2 (k+p_1+p_3)^2 [(k-p_4)^2 - M^2]}$$

$$I_{\triangleleft}^{t} \equiv \int \frac{d^d k}{i(2\pi)^d} \frac{1}{k^2 (k+p_2+p_4)^2 [(k+p_2)^2 - M^2]}$$

Massive Triangle

$$I_{\triangleright}^{t} \equiv \int \frac{d^d k}{i(2\pi)^d} \frac{1}{k^2 (k+p_1+p_3)^2 (k+p_1)^2}$$

Massless Triangle

$$I_{\circ}^{t} \equiv \int \frac{d^d k}{i(2\pi)^d} \frac{1}{k^2 (k+p_1+p_3)^2} \,.$$

Bubble

One-loop amplitudes

Master-integral decomposition

Same master integrals and coefficients as on the previous slide

$$D_t^{hh'} = \mathrm{Disc}_t \left\{ C_{\square}^{hh'} I_{\square}^{st} + C_{x\square}^{hh'} I_{\square}^{tu} + C_{\triangleleft}^{hh'} I_{\triangleleft}^{t} + C_{\triangleright}^{hh'} I_{\triangleright}^{t} + C_{\circ}^{hh'} I_{\circ}^{t} \right\}$$

Classical sector

$C_{\square}^{hh'} I_{\square}^{st} + C_{x\square}^{hh'} I_{\square}^{tu}$

Box topologies: imaginary and IR divergent

$C_{\triangleleft}^{hh'} I_{\triangleleft}^{t}$ Massive triangle: real, long-range

Quantum sector

$C_{\triangleright}^{hh'} I_{\triangleright}^{t}$

$C_{\circ}^{hh'} I_{\circ}^{t}$

Massless triangle and bubble

Consistency checks in the long-range potential

$\boldsymbol{q}^2 = 4\omega^2 \quad \Longleftrightarrow \quad \cos\theta = -1$ Spurious backward-scattering poles cancel.

$\text{Box terms} \sim \dfrac{i}{\boldsymbol{q}^2}\left(\dfrac{1}{\epsilon} - \log\dfrac{\boldsymbol{q}^2}{\mu^2}\right) \times (\ldots)$

Imaginary and IR divergent;
cancel against iteration.

One-loop amplitudes

One-loop amplitudes in Born regime:

$$\mathscr{M}^{(1)}[1^-_X 2_\Phi \to (-3)^-_X (-4)_\Phi] = \frac{M^3\langle 1|\sigma^0|3]^{2J}}{2M_{\rm Pl}^4(2\omega)^{2J}}\left\{ c_\square I_0 - \frac{c_\triangleleft}{2}\tilde{I}_0 \right\}$$

$$\mathscr{M}^{(1)}[1^-_X 2_\Phi \to (-3)^+_X (-4)_\Phi] = \frac{M^3\langle 13\rangle^{2J}}{2M_{\rm Pl}^4}\left\{ \tilde{c}_\square I_0 - \frac{\tilde{c}_\triangleleft}{2}\tilde{I}_0 \right\}$$

Master integrals:

$$I_0 \equiv \mu^{2\epsilon}\int \frac{d^dl}{(2\pi)^d}\frac{1}{\left[\boldsymbol{l}^2-\boldsymbol{p}^2\right](\boldsymbol{l}-\boldsymbol{p})^2(\boldsymbol{l}-\boldsymbol{p}')^2} = -\frac{i}{8\pi\omega\boldsymbol{q}^2}\left[\frac{1}{\varepsilon}-\log\left(\frac{\boldsymbol{q}^2}{\mu^2}\right)\right]$$

$$\tilde{I}_0 \equiv \mu^{2\epsilon}\int \frac{d^dl}{(2\pi)^d}\frac{1}{(\boldsymbol{l}-\boldsymbol{p})^2(\boldsymbol{l}-\boldsymbol{p}')^2} = \frac{1}{8|\boldsymbol{q}|},$$

Coefficients for J=1 (electromagnetic wave) and J=2 (gravitational wave)

$$\mathbf{J=1}: \quad c_\square = 16\omega^4\frac{1-\frac{\boldsymbol{q}^2}{4\omega^2}+\frac{\boldsymbol{q}^4}{32\omega^4}}{\left[1-\frac{\boldsymbol{q}^2}{4\omega^2}\right]^2}, \qquad c_\triangleleft = -15\omega^2\frac{1-\frac{\boldsymbol{q}^2}{6\omega^2}+\frac{\boldsymbol{q}^4}{80\omega^4}}{\left[1-\frac{\boldsymbol{q}^2}{4\omega^2}\right]^2} + \mathrm{Re}C_g\omega^2\boldsymbol{q}^4$$

$$\tilde{c}_\square = 16C_\gamma\omega^4, \quad \tilde{c}_\triangleleft = -15C_\gamma\omega^2\left(1-\frac{\boldsymbol{q}^2}{12\omega^2}\right)$$

$$\mathbf{J=2}: \quad c_\square = \frac{16\omega^4\left[1-\frac{\boldsymbol{q}^2}{4\omega^2}+\frac{\boldsymbol{q}^4}{16\omega^4}\right]^2+\mathrm{Re}C_g\boldsymbol{q}^8}{\left[1-\frac{\boldsymbol{q}^2}{4\omega^2}\right]^4} \qquad c_\triangleleft = \frac{-15\omega^2\left[1-\frac{7\boldsymbol{q}^2}{60\omega^2}-\frac{7\boldsymbol{q}^4}{240\omega^4}+\frac{\boldsymbol{q}^6}{64\omega^6}\right]+\mathrm{Re}C_g\boldsymbol{q}^4\omega^2\left[1-\frac{9\boldsymbol{q}^2}{4\omega^2}-\frac{9\boldsymbol{q}^4}{16\omega^4}+\frac{\boldsymbol{q}^6}{64\omega^6}\right]}{\left[1-\frac{\boldsymbol{q}^2}{4\omega^2}\right]^4}$$

$$\tilde{c}_\square = 1+16C_g\omega^4, \quad \tilde{c}_\triangleleft = -10C_g\omega^2\left(1-\frac{3\boldsymbol{q}^2}{20\omega^2}\right)$$

see backup for other spins

Iteration term

$$\mathcal{I}^{(1)}_{hh'} = \frac{(-)^{2J}}{2M}\int \frac{d^d\boldsymbol{l}}{(2\pi)^d}\frac{\mathscr{M}^{(0)}[1^h_X 2_\Phi \to l^a_X l'_\Phi]\mathscr{M}^{(0)}[l^a_X l'_\Phi \to (-3)^{h'}_X (-4)_\Phi]}{(\boldsymbol{l}^2-\boldsymbol{p}^2)}$$

After plugging tree-level amplitudes

$$\mathcal{I}^{(1)}_{--} = \frac{M^3\langle 1|\sigma^0|3]^{2J}}{2M_{\rm Pl}^4}\left\{\sum_{k=0}^{2J}(-)^k(2\omega)^{4-2J-k}\binom{2J}{k}\Lambda_k + \delta_{J2}\left[\frac{1}{(2\omega)^4}+2{\rm Re}C_X\right]\Lambda_4\right\}$$

$$\mathcal{I}^{(1)}_{-+} = \frac{M^3(2\omega)^{4-2J}\langle 13\rangle^{2J}}{M_{\rm Pl}^4}\left[\frac{\delta_{J2}}{(2\omega)^4}+C_X\right]\tilde{\Lambda}_{2J},$$

Integrals are defined by

$$\langle 1|\sigma^0|3]^n\Lambda_n \equiv \int\frac{d^d\boldsymbol{l}}{(2\pi)^d}\frac{\langle 1|l|3]^n}{(\boldsymbol{l}^2-\boldsymbol{p}^2)(\boldsymbol{l}-\boldsymbol{p})^2(\boldsymbol{l}-\boldsymbol{p}')^2}$$

$$\langle 13\rangle^n\tilde{\Lambda}_n \equiv \int\frac{d^d\boldsymbol{l}}{(2\pi)^d}\frac{\langle 1|\sigma^0 l|3\rangle^n}{(\boldsymbol{l}^2-\boldsymbol{p}^2)(\boldsymbol{l}-\boldsymbol{p})^2(\boldsymbol{l}-\boldsymbol{p}')^2}\,.$$

and are functions of master integrals $I_0, \tilde{I}_0$ with imaginary part and spurious poles

One-loop potential

Subtract iteration term to obtain potential:

$$V^{(1)h}_{\ \ h'}(\boldsymbol{p},\boldsymbol{p}') = \mathscr{M}^{(1)h}_{\ \ h'} - \mathscr{I}^{(1)h}_{\ \ h'}$$

When dust settles, imaginary parts and spurious poles magically cancel !

$$\mathbf{J=1}: \quad V^{(1)} = \begin{pmatrix} \frac{15M^3\omega^2}{32M_{\rm Pl}^4|\boldsymbol{q}|}\left(1-\frac{7\boldsymbol{q}^2}{60\omega^2}-\frac{\boldsymbol{q}^4}{96\omega^4}-\mathrm{Re}C_g\frac{\boldsymbol{q}^4}{15}\right) & -\frac{9C_\gamma M^3\omega^2\boldsymbol{q}_-^2}{32M_{\rm Pl}^4|\boldsymbol{q}|}\left(1+\frac{7\boldsymbol{q}^2}{18\omega^2}\right) \\ -\frac{9C_\gamma^* M^3\omega^2\boldsymbol{q}_+^2}{32M_{\rm Pl}^4|\boldsymbol{q}|}\left(1+\frac{7\boldsymbol{q}^2}{18\omega^2}\right) & \frac{15M^3\omega^2}{32M_{\rm Pl}^4|\boldsymbol{q}|}\left(1-\frac{7\boldsymbol{q}^2}{60\omega^2}-\frac{\boldsymbol{q}^4}{96\omega^4}-\mathrm{Re}C_g\frac{\boldsymbol{q}^4}{15}\right) \end{pmatrix} + \mathcal{O}(|\boldsymbol{q}|^5)$$

$$\mathbf{J=2}: \quad V^{(1)} = \begin{pmatrix} \frac{15M^3\omega^2}{32M_{\rm Pl}^4|\boldsymbol{q}|}\left(1-\frac{\boldsymbol{q}^2}{60\omega^2}-\frac{\boldsymbol{q}^4}{30\omega^4}-\mathrm{Re}C_g\frac{\boldsymbol{q}^4}{15}\right) & -\frac{25M^3\omega^2\boldsymbol{q}_-^4}{32M_{\rm Pl}^4|\boldsymbol{q}|}\left(C_g+\frac{7}{80\omega^4}\right) \\ -\frac{25M^3\omega^2\boldsymbol{q}_+^4}{32M_{\rm Pl}^4|\boldsymbol{q}|}\left(C_g^*+\frac{7}{80\omega^4}\right) & \frac{15M^3\omega^2}{32M_{\rm Pl}^4|\boldsymbol{q}|}\left(1-\frac{\boldsymbol{q}^2}{60\omega^2}-\frac{\boldsymbol{q}^4}{30\omega^4}-\mathrm{Re}C_g\frac{\boldsymbol{q}^4}{15}\right) \end{pmatrix} + \mathcal{O}(|\boldsymbol{q}|^5)$$

Suppressed by $\dfrac{M|\boldsymbol{q}|}{M_{\rm Pl}^2} \sim |\boldsymbol{q}|R_s \sim R_s/b$ **compared to tree level**

At 1 loop, qualitatively new beyond GR effects of $\sim \mathrm{Re}C_g$ universal helicity-conserving corrections

One-loop potential

$$\mathbf{J=0}: \quad V^{(1)}=\frac{15M^3\omega^2}{32M_{\rm Pl}^4|\boldsymbol{q}|}\left(1-\frac{\boldsymbol{q}^2}{60\omega^2}-\mathrm{Re}C_g\frac{\boldsymbol{q}^4}{15}\right)+\mathcal{O}(|\boldsymbol{q}|^5)$$

$$\mathbf{J=\frac{1}{2}}: \quad V^{(1)}=\frac{15M^3\omega^2}{32M_{\rm Pl}^4|\boldsymbol{q}|}\left(1-\frac{11\boldsymbol{q}^2}{120\omega^2}-\frac{7\boldsymbol{q}^4}{960\omega^4}-\mathrm{Re}C_g\frac{\boldsymbol{q}^4}{15}\right)\begin{pmatrix}1&0\\0&1\end{pmatrix}+\mathcal{O}(|\boldsymbol{q}|^5)$$

$$\mathbf{J=1}: \quad V^{(1)}=\begin{pmatrix}\frac{15M^3\omega^2}{32M_{\rm Pl}^4|\boldsymbol{q}|}\left(1-\frac{7\boldsymbol{q}^2}{60\omega^2}-\frac{\boldsymbol{q}^4}{96\omega^4}-\mathrm{Re}C_g\frac{\boldsymbol{q}^4}{15}\right) & -\frac{9C_\gamma M^3\omega^2\boldsymbol{q}_-^2}{32M_{\rm Pl}^4|\boldsymbol{q}|}\left(1+\frac{7\boldsymbol{q}^2}{18\omega^2}\right)\\ -\frac{9C_\gamma^* M^3\omega^2\boldsymbol{q}_+^2}{32M_{\rm Pl}^4|\boldsymbol{q}|}\left(1+\frac{7\boldsymbol{q}^2}{18\omega^2}\right) & \frac{15M^3\omega^2}{32M_{\rm Pl}^4|\boldsymbol{q}|}\left(1-\frac{7\boldsymbol{q}^2}{60\omega^2}-\frac{\boldsymbol{q}^4}{96\omega^4}-\mathrm{Re}C_g\frac{\boldsymbol{q}^4}{15}\right)\end{pmatrix}$$
$$+\mathcal{O}(|\boldsymbol{q}|^5)$$

$$\mathbf{J=\frac{3}{2}}: \quad V^{(1)}=\frac{15M^3\omega^2}{32M_{\rm Pl}^4|\boldsymbol{q}|}\left(1-\frac{11\boldsymbol{q}^2}{120\omega^2}-\frac{\boldsymbol{q}^4}{64\omega^4}\right)\begin{pmatrix}1&0\\0&1\end{pmatrix}+\mathcal{O}(|\boldsymbol{q}|^5)$$

$$\mathbf{J=2}: \quad V^{(1)}=\begin{pmatrix}\frac{15M^3\omega^2}{32M_{\rm Pl}^4|\boldsymbol{q}|}\left(1-\frac{\boldsymbol{q}^2}{60\omega^2}-\frac{\boldsymbol{q}^4}{30\omega^4}-\mathrm{Re}C_g\frac{\boldsymbol{q}^4}{15}\right) & -\frac{25M^3\omega^2\boldsymbol{q}_-^4}{32M_{\rm Pl}^4|\boldsymbol{q}|}\left(C_g+\frac{7}{80\omega^4}\right)\\ -\frac{25M^3\omega^2\boldsymbol{q}_+^4}{32M_{\rm Pl}^4|\boldsymbol{q}|}\left(C_g^*+\frac{7}{80\omega^4}\right) & \frac{15M^3\omega^2}{32M_{\rm Pl}^4|\boldsymbol{q}|}\left(1-\frac{\boldsymbol{q}^2}{60\omega^2}-\frac{\boldsymbol{q}^4}{30\omega^4}-\mathrm{Re}C_g\frac{\boldsymbol{q}^4}{15}\right)\end{pmatrix}$$
$$+\mathcal{O}(|\boldsymbol{q}|^5)\,.$$

Applications to lensing beyond GR

Applications

GW lensing

- GW lensing happens in triple systems

  Also through intervening galaxies, clusters, compact objects...

  ⇨ One of the goals of the LISA mission

  Einstein Telescope should see hundreds

- Standard signature of GW lensing

  ⇨ Multiple copies of the same GW

  Also, frequency-dependent amplification and interference without separately resolved images

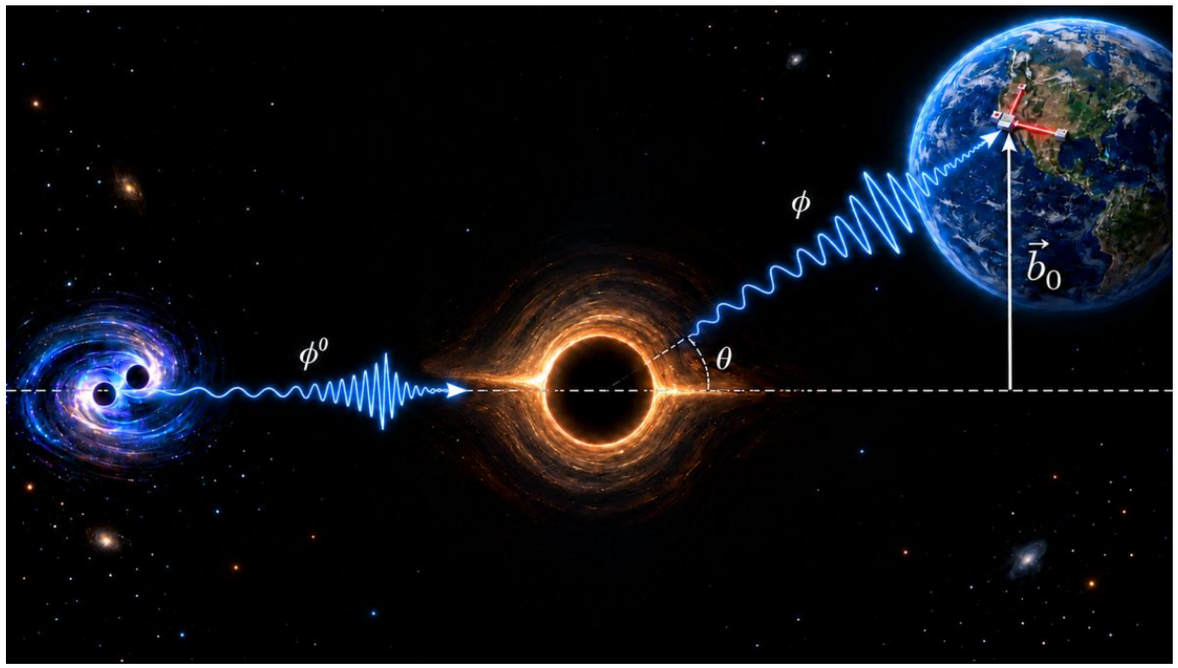
- Wave optics (Born limit) effects are expected to appear in LISA

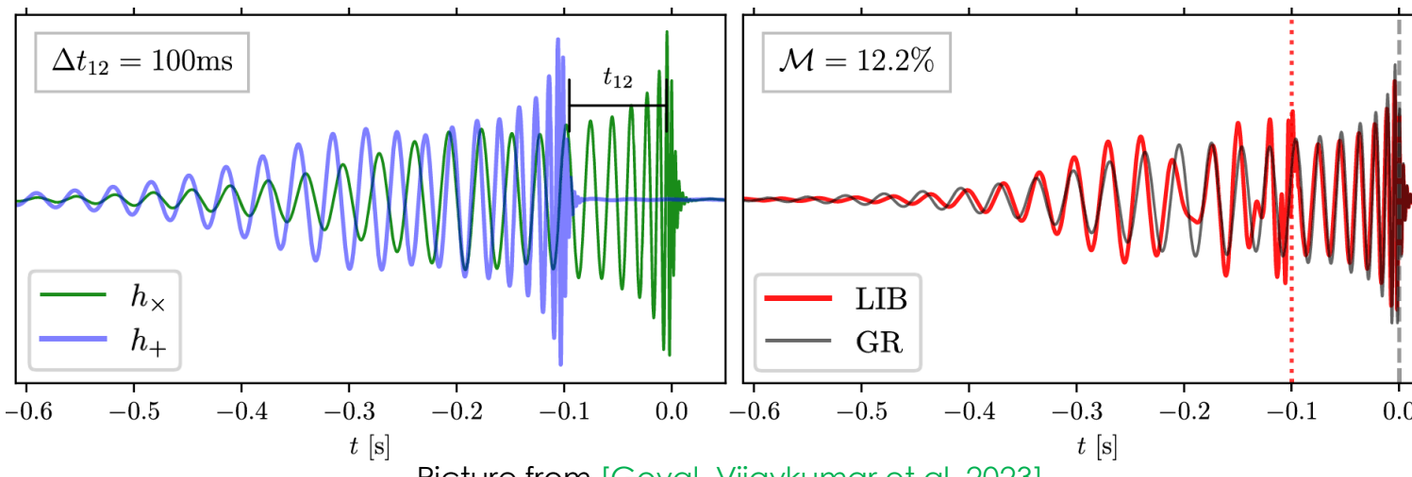
  [Pijnenburg, Cusin, Pitrou, Uzan, 2024]

- Beyond GR effects in GW lensing

  ⇨ Time delay of the +,x polarizations

  [Ezquiaga, Zumalacárregui 2020] [Goyal, Vijaykumar et al. 2023]





Picture from [Goyal, Vijaykumar et al. 2023]

From scattering potential to diffraction

The same 2PM potential extracted from flat-space amplitudes determines the lensing phase.

1. **On-shell amplitude** — Tree + one-loop
2. **LS matching** — Subtract iterations
3. **Time-delay phase** — Impact-parameter space
4. **Lensed signal** — Matrix diffraction

Potential → gravitational time delay

$$\psi^{hh\prime}(\boldsymbol{b}) = \frac{1}{2\omega^2 D}\int dz\, U^{hh\prime}(\boldsymbol{b}, z)$$

$$\psi(\boldsymbol{b}) = -\frac{1}{4M\omega^2 D}\int \frac{d^2\tilde{\boldsymbol{q}}}{(2\pi)^2}\, e^{i\tilde{\boldsymbol{q}}\cdot\boldsymbol{b}}\, \boldsymbol{V}(\boldsymbol{p}, \boldsymbol{p\prime})\Big|_{q_z=0}$$

Diffraction integral (matrix-valued)

$$\boldsymbol{F}(\boldsymbol{\theta}_*) = \frac{\omega D}{2\pi i}\int d^2\boldsymbol{\theta}\; e^{i\omega D|\boldsymbol{\theta}-\boldsymbol{\theta}_*|^2/2}\; e^{-i\omega D\boldsymbol{\psi}(D_l\boldsymbol{\theta})}$$

$D \equiv \dfrac{D_l D_*}{D_{l*}}, \qquad \boldsymbol{b} = D_l\boldsymbol{\theta}$

Matrix exponential in the eikonal / time-delay approximation.

Nakamura & Deguchi (1999); Carrillo González et al. (arXiv:2511.15797).

Applications

Helicity structure of the lensing potential

The potential is a 2 × 2 matrix in the helicity basis for X = γ (J = 1) and X = g (J = 2).

$$\boldsymbol{V}_X = V_{\mathrm{d}}^X\,\mathbf{1} + \begin{pmatrix} 0 & V_{\mathrm{f}}^X \\ (V_{\mathrm{f}}^X)^* & 0 \end{pmatrix}$$

PHOTON · J = 1

Minimal GR preserves photon helicity in the displayed classical potential.

$$V_{\mathrm{f}}^{\gamma} \propto C_\gamma \qquad (J=1)$$

The RF² interaction generates helicity-changing scattering.

GRAVITON · J = 2

Einstein–Hilbert gravity already produces a helicity-flip channel.

$$V_{\mathrm{f}}^{g} = V_{\mathrm{f}}^{g,\mathrm{EH}} + C_g\,\Delta V_{\mathrm{f}}^{g,R^3} \qquad (J=2)$$

The R³ interaction modifies both diagonal and flip terms.

$$V_{\mathrm{d}}^X = V_{\mathrm{GR}}^X + \mathrm{Re}\,C_g\,\Delta V_{R^3}^X + \cdots$$

Helicity flipping effects can enhance sensitivity to new interactions even if absolute size of new physics is small !

Matrix diffraction: eigenphases and mixing

For parity-even couplings, the local 2 × 2 phase matrix has two eigenvalues.

Impact-parameter phase

$$\boldsymbol{\psi}_X(b,\varphi_b) = \psi_{\mathrm{d}}^X(b)\,\mathbf{1} + \psi_{\mathrm{f}}^X(b) \begin{pmatrix} 0 & e^{-2iJ\varphi_b} \\ e^{+2iJ\varphi_b} & 0 \end{pmatrix} \leftarrow S_J$$

PROPAGATION EIGENPHASES

$$\psi_{\pm}^X(b) = \psi_{\mathrm{d}}^X(b) \pm \psi_{\mathrm{f}}^X(b)$$

$$e^{-i\omega D\boldsymbol{\psi}_X} = e^{-i\omega D\psi_{\mathrm{d}}^X}\left[\cos(\omega D\psi_{\mathrm{f}}^X)\,\mathbf{1} - i\sin(\omega D\psi_{\mathrm{f}}^X)\,\boldsymbol{S}_J\right]$$

Helicity conserving

$$\left[e^{-i\omega D\boldsymbol{\psi}_X}\right]_{hh} = e^{-i\omega D\psi_{\mathrm{d}}^X}\cos(\omega D\psi_{\mathrm{f}}^X)$$

Helicity changing

$$\left[e^{-i\omega D\boldsymbol{\psi}_X}\right]_{h,-h} \propto e^{-i\omega D\psi_{\mathrm{d}}^X}\sin(\omega D\psi_{\mathrm{f}}^X)\,e^{\pm 2iJ\varphi_b}$$

IMPORTANT The angle-dependent mixing matrix stays inside the diffraction integral.

From helicities to linear polarizations

Choose the polarization axes along the lens–source plane (parity-even, axisymmetric setup).

$$\begin{pmatrix} h_+ \\ h_\times \end{pmatrix} = \boldsymbol{U} \begin{pmatrix} h_{+2} \\ h_{-2} \end{pmatrix}, \qquad \boldsymbol{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

The observable amplification matrix is diagonal in this adapted basis:

$$\boldsymbol{F}_{\rm lin} = \boldsymbol{U}\boldsymbol{F}_{\rm hel}\boldsymbol{U}^\dagger \simeq \begin{pmatrix} F_+ & 0 \\ 0 & F_\times \end{pmatrix}$$

Geometric-optics limit: sum over stationary images

$$F_a(\omega) \simeq \sum_j \sqrt{|\mu_j|}\, e^{i\omega\tau_{a,j} - i\pi n_j} \qquad (a = +, \times)$$

$$\tau_{a,j} \simeq t_{{\rm geo},j} - D\left[\psi_{\rm d}(b_j) + s_a \psi_{\rm f}(b_j)\right], \quad s_+ = +1,\ s_\times = -1$$

Delays shown at first order in the polarization splitting; the Morse phase is included in n_j.

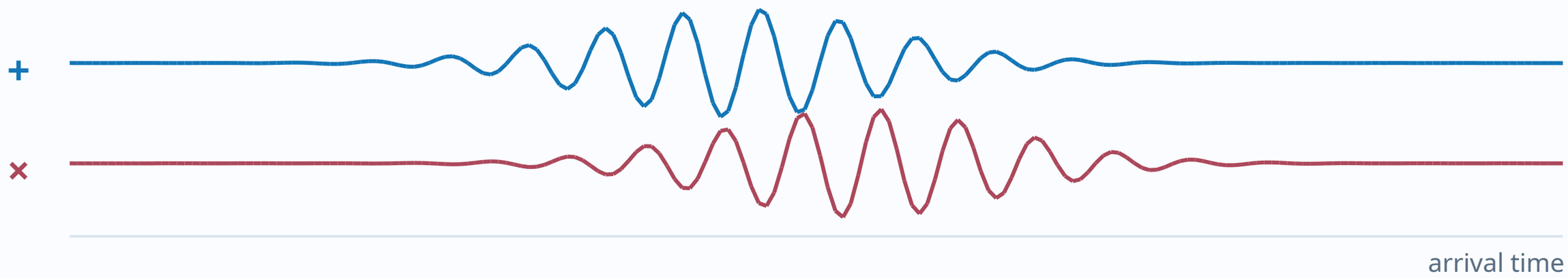
Observable signal: polarization-dependent delay

Helicity-changing lensing produces distinct amplification phases for + and × polarizations.

$$\boldsymbol{h}_{\rm in} = \begin{pmatrix} A_+ \\ A_\times \end{pmatrix}, \qquad \boldsymbol{h}_{\rm out} = \begin{pmatrix} F_+ A_+ \\ F_\times A_\times \end{pmatrix}$$

In the nondispersive approximation, each image shifts the two waveforms

$$\widetilde{\boldsymbol{h}}_{\rm out}(t) \simeq \sum_j \sqrt{|\mu_j|} \begin{pmatrix} \widetilde{A}_+(t-\tau_{+,j}) \\ \widetilde{A}_\times(t-\tau_{\times,j}) \end{pmatrix}$$



$$\Delta t_j \equiv \tau_{\times,j} - \tau_{+,j} \simeq 2D\psi_{\rm f}(b_j)$$

For dispersive phases, use group delay: $\Delta t \simeq 2\,\partial\omega[\omega\, D\, \psi f(\omega, bj)]$.

Related signatures: Ezquiaga & Zumalacárregui (2020); Goyal et al. (2023).

How large can the helicity-flip R^3 effect be?

Einstein–Hilbert gravity and the R^3 operator both contribute at order b^{-4}, but with different frequency dependence.

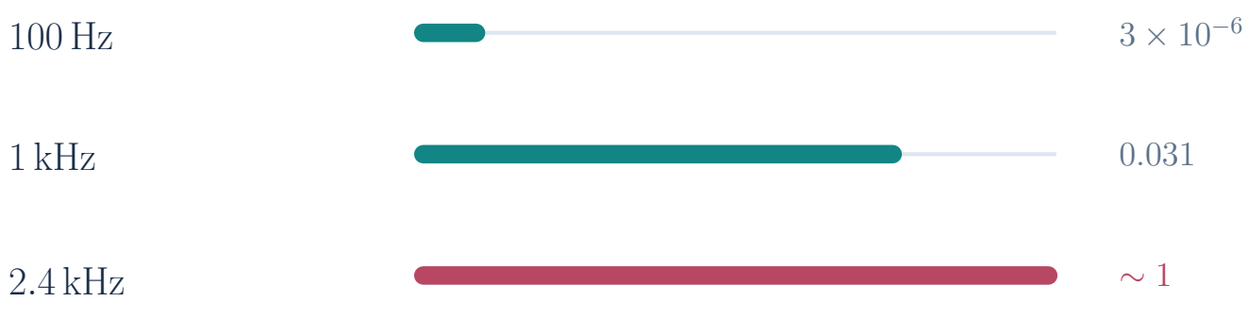
Graviton helicity-changing channel ($J = 2$)

$$\frac{\left|\psi^{g}_{\mathrm{f},R^3}\right|}{\left|\psi^{g}_{\mathrm{f},EH}\right|} = 16\,\omega^4|C_g| = 16\,(\omega\ell_g)^4$$

Define the higher-derivative scale by $|C_g| = \ell_g^4$, with $\omega = 2\pi f$.

Numerical benchmark: $\ell_g = 10\,\mathrm{km}$

Logarithmic guide: R^3 / EH flip contribution



What this comparison means

R^3 may compete with *the GR flip term*, not with the universal helicity-conserving lensing phase.

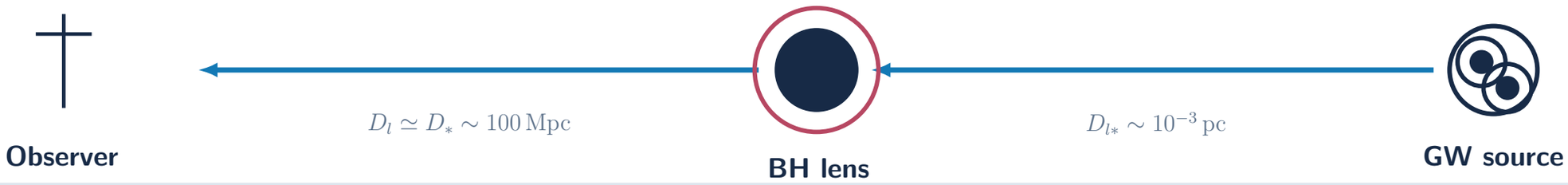
The formal crossover at $\omega\ell_g \simeq \frac{1}{2}$ is not parametrically within the low-energy EFT regime.

EFT CAUTION For a controlled low-energy expansion one requires $\omega\ell_g \ll 1$; an order-one ratio is not automatically a reliable prediction.

GW lensing: an illustrative R^3 time delay

A large relative correction in the flip channel can still imply an extraordinarily small absolute observable.

Triple-black-hole lensing geometry (not to scale)



Leading R^3 polarization delay at $b_E = D_l\theta_E$

$$\left|\Delta t^{(g)}_{\mathrm{pol},R^3}\right| = \frac{48\,|C_g|D^2}{R_s D_l^4} \simeq \frac{48\,|C_g|}{R_s D_{l*}^2}$$

BENCHMARK

$$\left|\Delta t^{(g)}_{\mathrm{pol},R^3}\right| \sim 6\times 10^{-30}\ \mathrm{s}$$

$\ell_g = 10\,\mathrm{km}$ ($|C_g| = \ell_g^4$), $M = 10^8 M_\odot$ ($R_s \simeq 3\times 10^8\,\mathrm{km}$), $D_{l*} = 10^{-3}\,\mathrm{pc}$.

Relative sensitivity to R^3 does not imply a detectable time delay for this geometry.

Applications

Photon lensing: a clean polarization test

At the order studied, minimal GR preserves photon helicity, while the RF^2 operator generates a splitting of linear polarizations.

Photon ($J = 1$): helicity basis

$$\psi_{\mathrm{f}}^{\gamma,\mathrm{GR}}(b) = 0$$

$$|\psi_{\mathrm{f}}^{\gamma,RF^2}(b)| = \frac{M|C_\gamma|}{\pi M_{\mathrm{Pl}}^2 D b^2}$$

Adapted linear-polarization basis

$$\psi_{x,y}^{\gamma} = \psi_{\mathrm{d}}^{\gamma} \pm \psi_{\mathrm{f}}^{\gamma}$$

The two modes acquire different phases, even when their total magnification differs only slightly.

Relative arrival time at the reference Einstein-ring image

$$\left|\Delta t_{\mathrm{pol}}^{(\gamma)}(b_E)\right| = \frac{4\,|C_\gamma| D}{D_l^2}$$

$$b_E = D_l \theta_E, \qquad \theta_E^2 = \frac{4GM}{D}, \qquad D = \frac{D_l D_*}{D_{l*}}$$

Leading C_γ contribution, geometric-optics saddle; polarization axes adapted to the lens plane.

Conclusions

1. Modern amplitude methods are useful for calculating classical effects in gravitational theories
2. Systematic approach to calculate wave-optics and post-Minkowskian (PM) corrections
3. Straightforward to calculate effects beyond general relativity
4. We calculated effects from leading higher-dimensional operators added to Einstein-Hilbert-Maxwell Lagrangian through 2PM order
5. Interesting observables include polarization-dependent time delay of electromagnetic and gravitational waves

Outlook

1. Comprehensive phenomenological analysis
   of FFR and R^3 effects on lensing

2. Lensing in UV completions of R^3 theories

3. Other deformation of GR,
   in particular scalar-tensor gravity

4. Including spin of massive scatterer

THANK YOU



Back up

One-loop amplitudes

Amplitudes in Born regime:

$$\mathscr{M}^{(1)}[1^-_X 2_\Phi \to (-3)^-_X (-4)_\Phi] = \frac{M^3\langle 1|\sigma^0|3]^{2J}}{2M_{\rm Pl}^4(2\omega)^{2J}}\left\{ c_\square I_0 - \frac{c_\triangleleft}{2}\tilde{I}_0 \right\}$$

$$\mathscr{M}^{(1)}[1^-_X 2_\Phi \to (-3)^+_X (-4)_\Phi] = \frac{M^3\langle 13\rangle^{2J}}{2M_{\rm Pl}^4}\left\{ \tilde{c}_\square I_0 - \frac{\tilde{c}_\triangleleft}{2}\tilde{I}_0 \right\}$$

$$I_0 \equiv \mu^{2\epsilon}\int \frac{d^dl}{(2\pi)^d}\frac{1}{\left[\boldsymbol{l}^2-\boldsymbol{p}^2\right](\boldsymbol{l}-\boldsymbol{p})^2(\boldsymbol{l}-\boldsymbol{p}')^2} = -\frac{i}{8\pi\omega\boldsymbol{q}^2}\left[\frac{1}{\epsilon}-\log\left(\frac{\boldsymbol{q}^2}{\mu^2}\right)\right]$$

$$\tilde{I}_0 \equiv \mu^{2\epsilon}\int \frac{d^dl}{(2\pi)^d}\frac{1}{(\boldsymbol{l}-\boldsymbol{p})^2(\boldsymbol{l}-\boldsymbol{p}')^2} = \frac{1}{8|\boldsymbol{q}|},$$

$$\mathbf{J=0}: \quad c_\square = 16\omega^4, \quad c_\triangleleft = -15\omega^2\left(1-\frac{\boldsymbol{q}^2}{60\omega^2}\right) + \mathrm{Re}C_g\omega^2\boldsymbol{q}^4\left(1+\frac{\boldsymbol{q}^2}{4\omega^2}\right), \quad \tilde{c}_\square = \tilde{c}_\triangleleft = 0$$

$$\mathbf{J=\frac{1}{2}}: \quad c_\square = 16\omega^4\frac{1-\frac{\boldsymbol{q}^2}{8\omega^2}}{1-\frac{\boldsymbol{q}^2}{4\omega^2}}, \quad c_\triangleleft = -15\omega^2\frac{1-\frac{7\boldsymbol{q}^2}{60\omega^2}}{1-\frac{\boldsymbol{q}^2}{4\omega^2}} + \mathrm{Re}C_g\omega^2\boldsymbol{q}^4, \quad \tilde{c}_\square = \tilde{c}_\triangleleft = 0$$

$$\mathbf{J=1}: \quad c_\square = 16\omega^4\frac{1-\frac{\boldsymbol{q}^2}{4\omega^2}+\frac{\boldsymbol{q}^4}{32\omega^4}}{\left[1-\frac{\boldsymbol{q}^2}{4\omega^2}\right]^2}, \qquad c_\triangleleft = -15\omega^2\frac{1-\frac{\boldsymbol{q}^2}{6\omega^2}+\frac{\boldsymbol{q}^4}{80\omega^4}}{\left[1-\frac{\boldsymbol{q}^2}{4\omega^2}\right]^2} + \mathrm{Re}C_g\omega^2\boldsymbol{q}^4$$

$$\tilde{c}_\square = 16C_\gamma\omega^4, \quad \tilde{c}_\triangleleft = -15C_\gamma\omega^2\left(1-\frac{\boldsymbol{q}^2}{12\omega^2}\right)$$

$$\mathbf{J=\frac{3}{2}}: \quad c_\square = 16\omega^4\frac{1-\frac{3\boldsymbol{q}^2}{8\omega^2}+\frac{3\boldsymbol{q}^4}{32\omega^4}-\frac{\boldsymbol{q}^6}{128\omega^6}}{\left[1-\frac{\boldsymbol{q}^2}{4\omega^2}\right]^3}, \qquad c_\triangleleft = -15\omega^2\frac{1-\frac{\boldsymbol{q}^2}{6\omega^2}+\frac{\boldsymbol{q}^4}{80\omega^4}}{\left[1-\frac{\boldsymbol{q}^2}{4\omega^2}\right]^3} \quad \tilde{c}_\square = \tilde{c}_\triangleleft = 0$$

$$\mathbf{J=2}: \quad c_\square = \frac{16\omega^4\left[1-\frac{\boldsymbol{q}^2}{4\omega^2}+\frac{\boldsymbol{q}^4}{16\omega^4}\right]^2 + \mathrm{Re}C_g\boldsymbol{q}^8}{\left[1-\frac{\boldsymbol{q}^2}{4\omega^2}\right]^4}$$

$$c_\triangleleft = \frac{-15\omega^2\left[1-\frac{7\boldsymbol{q}^2}{60\omega^2}-\frac{7\boldsymbol{q}^4}{240\omega^4}+\frac{\boldsymbol{q}^6}{64\omega^6}\right] + \mathrm{Re}C_g\boldsymbol{q}^4\omega^2\left[1-\frac{9\boldsymbol{q}^2}{4\omega^2}-\frac{9\boldsymbol{q}^4}{16\omega^4}+\frac{\boldsymbol{q}^6}{64\omega^6}\right]}{\left[1-\frac{\boldsymbol{q}^2}{4\omega^2}\right]^4}$$

$$\tilde{c}_\square = 1+16C_g\omega^4, \quad \tilde{c}_\triangleleft = -10C_g\omega^2\left(1-\frac{3\boldsymbol{q}^2}{20\omega^2}\right)$$