

Spectrum of functional renormalization

Paweł Jakubczyk
IFT FUW

1 SCALAR ϕ^4 MODEL

$$\mathcal{H}[\phi] = \int_V d^d x \left[\frac{1}{2} (\nabla \phi)^2 + \mathcal{U}(\phi^2) \right]$$

STATISTICAL
PHYSICS

$$\hookrightarrow Z = \int \mathcal{D}\phi e^{-\mathcal{H}[\phi]}$$

$$\phi \in \mathbb{R} \quad \xi := \frac{1}{2} \phi^2$$
$$\mathcal{U}(\xi) = \frac{\lambda}{2} (\xi - \xi_0)^2$$



- ϕ SCALAR (FOR SIMPLICITY)
- d ARBITRARY, LATER SPECIFY TO $d=3$
- $k_B T = 1$, UNITS CHOSEN SO THAT COEFFICIENT OF $(\nabla \phi)^2$ IS $\frac{1}{2}$
- $V = L^d$, PBC ON $\phi(\vec{x})$, THERMODYNAMIC LIMIT ($V \rightarrow \infty$)
- COARSE-GRAINED MODEL $\rightarrow$ UPPER MOMENTUM CUTOFF IMPLICIT.
- "PROCEDURAL DEFINITION" OF $\mathcal{D}\phi$, E.G. $\mathcal{D}\phi = \prod_{\vec{q}: |\vec{q}| < \Lambda_0} d\phi_{\vec{q}}$
(+ CONSTRAINTS FROM $\phi \in \mathbb{R}$)

2 THE CRITICAL POINT (d=3)

$$\xi := \frac{1}{2} \varphi^2 \quad \mathcal{U}(\xi) = \frac{\lambda}{2} (\xi - \xi_0)^2$$

- LOW T (LARGE ξ_0) $\rightarrow$ FERROMAGNETIC PHASE (SPONTANEOUSLY BROKEN $\mathbb{Z}_2$ SYMMETRY)
- HIGH T (SMALL ξ_0) $\rightarrow$ PARAMAGNETIC PHASE
- $T = T_c$ ($\xi = \xi_0^c$) $\rightarrow$ IIND ORDER PHASE TRANSITION (CRITICAL POINT)

} P.T. "DIVERGES" AT T_c }
{ (INFRARED PROBLEM) }

(SINGULARITIES OF THERMODYNAMIC PROPERTIES AND CORRELATION FUNCTIONS)

$$\text{E.G. } \otimes \langle \varphi(\vec{x}) \varphi(\vec{x}') \rangle \sim e^{-\frac{|\vec{x} - \vec{x}'|}{\xi}}$$

$$T \rightarrow T_c^+$$

$$\otimes \langle \varphi(\vec{x}) \varphi(\vec{x}') \rangle \stackrel{T=T_c}{\sim} \frac{1}{|\vec{x} - \vec{x}'|^{d-2+\eta}}$$

$$\xi \sim (T - T_c)^{-\nu}$$

- EXPONENTS ARE UNIVERSAL (HERE INDEPENDENT OF PARTICULAR FORM OF $\mathcal{U}(\xi)$) $\nu > 0$
 $\eta > 0$ (for $d=3$)

- EXPONENTS CAN BE COMPUTED FROM THE SPECTRUM OF RG TRANSFORMATION LINEARIZED AROUND THE RG FIXED POINT.

$$\text{IN PARTICULAR: } \nu = |e_1^{-1}|$$

- ONLY TWO EXPONENTS ARE INDEPENDENT

3d Ising

	ν	η
Conformal bootstrap	0.629 971(4)	0.036 297 8(20)
Six loop	0.6304(13)	0.0335(25)
High T	0.630 12(16)	0.036 39(15)
MC	0.630 02(10)	0.036 27(10)

Another case:

3d uniaxial Lifshitz

	θ	$\eta_{\perp}$	$\nu_{\perp}$	$\eta_{\parallel}$	$\nu_{\parallel}$	ϕ
MF	1/2	0	1/2	0	1/4	1/2
$\mathcal{O}(\epsilon)$ [38]	1/2	0	0.625	0	0.313	0.625
$\mathcal{O}(\epsilon^2)$ [38]	0.47	0.039	0.709	-0.019	0.348	0.677
Improved $\mathcal{O}(\epsilon^2)$ [39]	0.47	0.124	0.746	-0.019	0.348	0.715
1/N [40, 63]	0.451	0.306	0.726	0.223	0.266	-
LPA (Ref. [28])	1/2	0	0.63	0	-	-
DE(2)+$\mathbf{W}_{\parallel}(\rho)$	0.455±0.011	0.125±0.031	0.679±0.012	-0.12±0.03	0.3091± 0.0093	0.48±0.12

3 WETTERICH EQUATION

$$\bar{Z} = \int \mathcal{D}\phi e^{-\mathcal{H}[\phi]} \longrightarrow Z[J] = \int \mathcal{D}\phi e^{-\mathcal{H}[\phi] + J\phi} \quad \left(J\phi := \int_x J(x)\phi(x) \right)$$

$$W[J] = \log \bar{Z}[J] \quad (\text{HELMHOLTZ FREE ENERGY})$$

$W[J]$ GENERATES (CONNECTED) CORRELATION FUNCTIONS OF ϕ :

$$\bullet \frac{\delta W}{\delta J(x)} = \langle \phi(x) \rangle \stackrel{(x)}{=} \phi(x) \quad \bullet \frac{\delta^2 W}{\delta J(x) \delta J(y)} = \langle \phi(x)\phi(y) \rangle - \langle \phi(x) \rangle \langle \phi(y) \rangle$$

LEGENDRE TRANSFORM $\Gamma[\phi] := -W[J] + J\phi$ } $J[\phi(x)]$ FROM INVERTING (x)
 (GIBBS FREE ENERGY)

$$W[J] \longrightarrow W_k[J] = \log \int \mathcal{D}\phi e^{-\mathcal{H}[\phi] - \Delta\mathcal{H}_k[\phi] + J\phi}$$

MOMENTUM
CUTOFF SCALE

$$\Delta\mathcal{H}_k[\phi] := \frac{1}{2} \int_q \phi_{-q} R_k(q) \phi_q$$

IR REGULAR
FUNCTION

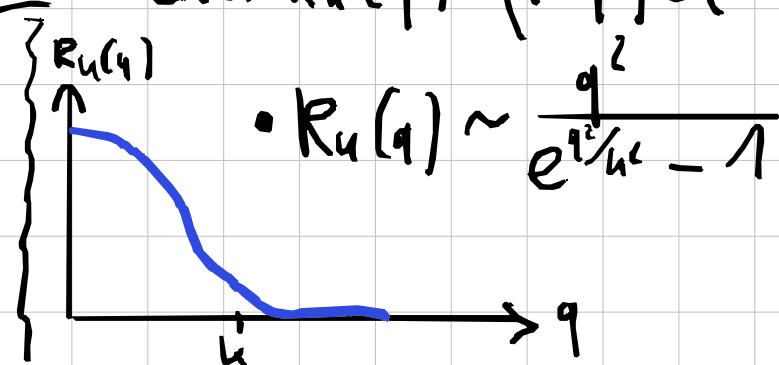
$\Delta\mathcal{H}_k[\phi]$ • GIVES AN EFFECTIVE MASS $m^2 \sim k^2$ TO MODES WITH $q^2 < k^2$
 • LEAVES MODES WITH $q^2 > k^2$ NOT AFFECTED

$$\text{E.G. } R_k(q) \sim (k^2 - q^2) \Theta(k^2 - q^2)$$

$$k \rightarrow 0 \quad W_k[J] \rightarrow W[J]$$

$$\phi(x) = \frac{\delta W_k[J]}{\delta J(x)}$$

$$\Gamma_k[\phi] := -W_k[J] + J\phi - \Delta\mathcal{H}_k[\phi]$$



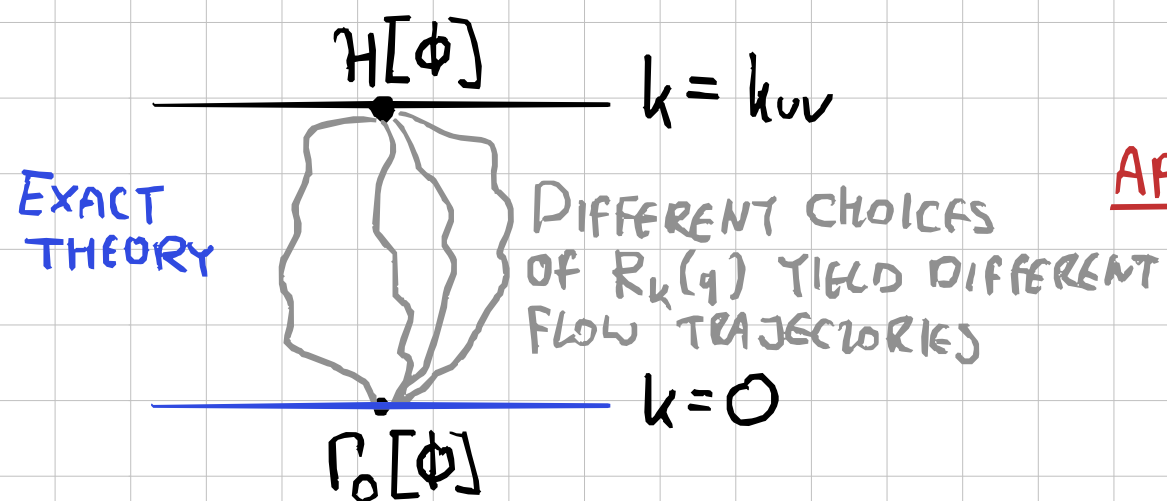
$\Gamma_k[\phi] \rightarrow$ GIBBS FREE ENERGY INCLUDING FLUCTUATION MODES WITH $q > k$

$$\Gamma_k[\phi] \begin{cases} \xrightarrow{k \rightarrow 0} \Gamma[\phi] & (\text{THERMODYNAMIC POTENTIAL}) \\ \xrightarrow{k \rightarrow \infty} \mathcal{H}[\phi] & (\text{"MICROSCOPIC" ACTION}) \end{cases}$$

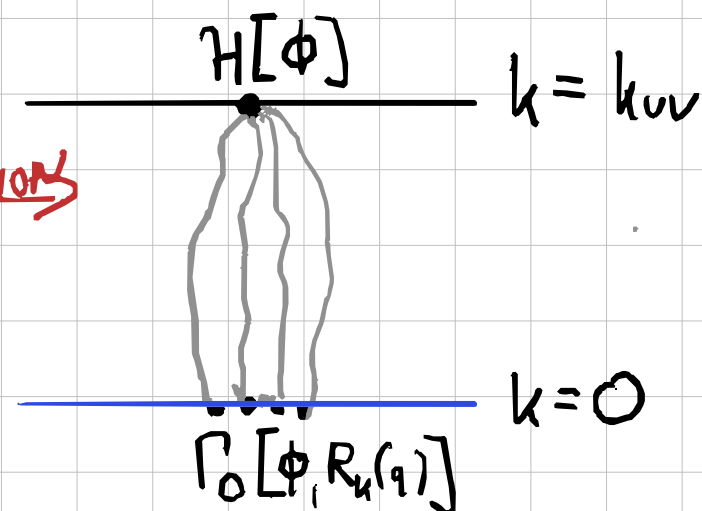
ACTING ON $\Gamma_k[\phi] := -W_k[J] + J\phi - \Delta \mathcal{H}_k[\phi]$ WITH ∂_k
GENERATES THE RG FLOW OF $\Gamma_k[\phi]$

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \left[(\partial_k R_k) (\Gamma^{(2)}[\phi] + R_k)^{-1} \right]$$

(WETTERICH EQ.)



APPROXIMATION



4 THE DERIVATIVE EXPANSION

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \left[\partial_k R_k (\Gamma^{(2)} + R_k)^{-1} \right]$$

• DE(0) (LPA) $\Gamma_k = \int_x \left[\mathcal{U}_k(\varphi) + \frac{1}{2} (\partial\phi)^2 \right]$ (MOMENTUM DEPENDENCIES NOT RENORMALIZED)

• DE(2) $\Gamma_k = \int_x \left[\mathcal{U}_k(\varphi) + \frac{1}{2} \bar{\mathcal{Z}}_k(\varphi) (\partial\phi)^2 \right]$ (TERMS OF ORDER ∂^4 NEGLECTED)

• DE(4) $\Gamma_k = \int_x \left[\mathcal{U}_k(\varphi) + \frac{1}{2} \bar{\mathcal{Z}}_k(\varphi) (\partial\phi)^2 + \frac{1}{2} \mathcal{W}_k^a(\varphi) (\partial_\mu \partial_\nu \phi)^2 + \frac{1}{2} \phi \mathcal{W}_k^b(\varphi) (\partial^2 \phi) (\partial\phi)^2 + \frac{1}{2} \mathcal{W}_k^c(\varphi) \left((\partial\phi)^2 \right)^2 \right]$

• DE(6) $\Gamma_k = \dots$

(SEQUENCE OF ANSÄTZE FOR $\Gamma_k[\phi]$)

3d Ising

Derivative expansion	ν	η
$s = 0$ (LPA)	0.651(1)	0
$s = 2$	0.6278(3)	0.0449 (6)
$s = 4$	0.630 39(18)	0.0343(7)
$s = 6$	0.630 12(5)	0.0361 (3)
$s \rightarrow \infty$	0.6300(2)	0.0358(6)
Conformal bootstrap	0.629 971(4)	0.036 297 8(20)

Balog *et al*, Phys. Rev. Lett. **123**, 240604 (2019)

Another case:

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5 DE(0) (LPA)

$$\Gamma_k = \int_x [\mathcal{U}_k(\varphi) + \frac{1}{2}(\partial\phi)^2]$$

$$\Gamma_k[\phi = \text{const}] = V \mathcal{U}_k(\varphi)$$

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \text{Tr} [\partial_k R_k (\Gamma^{(2)} + R_k)^{-1}]$$

$$\xrightarrow{\text{TAKE}} \left. \frac{\delta^2 \Gamma_k}{\delta\phi_{q_1} \delta\phi_{q_2}} \right|_{\phi = \text{const.}}$$

$$\partial_k \mathcal{U}_k(\varphi) = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q)}{q^2 + \mathcal{U}'_k(\varphi) + 2\varphi \mathcal{U}''_k(\varphi) + R_k(q)} \quad (\text{CLOSED PDE})$$

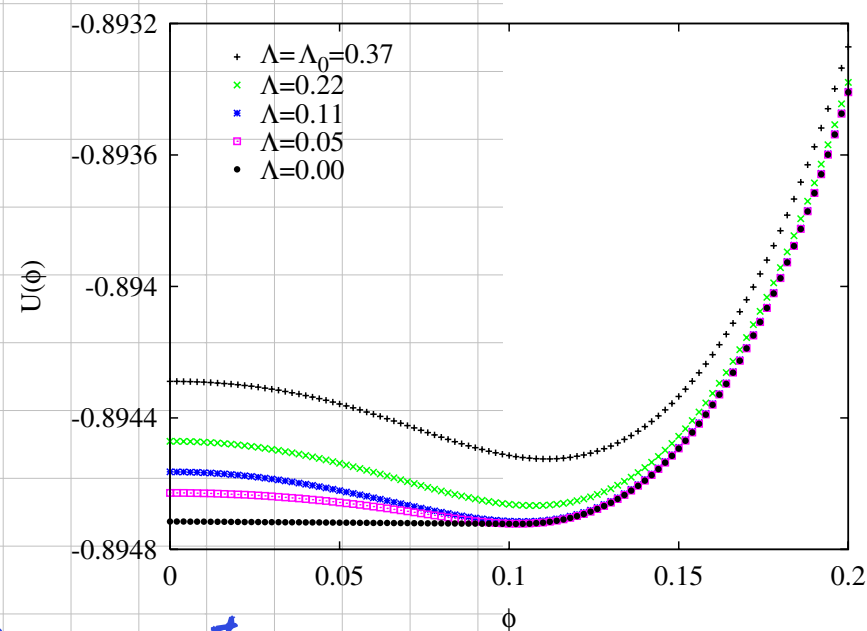
APPROACH TO CONVEXITY FOR $k \rightarrow 0$

(NEVER RECOVERED WITH FIELD-EXPANDED TREATMENT)

$$\text{RECALL: } \frac{\partial \mathcal{U}_{k=0}}{\partial \phi} = J \quad (\text{"MAGNETIC FIELD"})$$

$$J = 0^\pm \quad \text{FOR} \quad \begin{cases} \phi = 0^\pm & \text{HIGH-T PHASE} \\ \phi = \phi_0^\pm & \text{LOW-T PHASE} \end{cases}$$

(SPONTANEOUS SYMMETRY BREAKING)

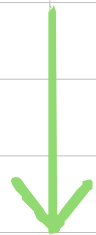


NONANALYTICITY OF $\mathcal{U}(\phi)$ BUILDS UP IN THE LIMIT $k \rightarrow 0^\pm$

6 DE(0) (LPA) — RG FIXED POINT

$$\partial_k U_k(\xi) = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q)}{q^2 + U_k(\xi) + 2\xi U_k''(\xi) + R_k(q)}$$

$$\left\{ \begin{array}{l} \tilde{\xi} = k^{2-d} \xi \quad y = \frac{q^2}{k^2} \quad u = U k^{-d} \quad r = R k^{-2} \quad t = \log \frac{k}{k_0} \\ + \text{ANGULAR INTEGRATION} \end{array} \right\}$$

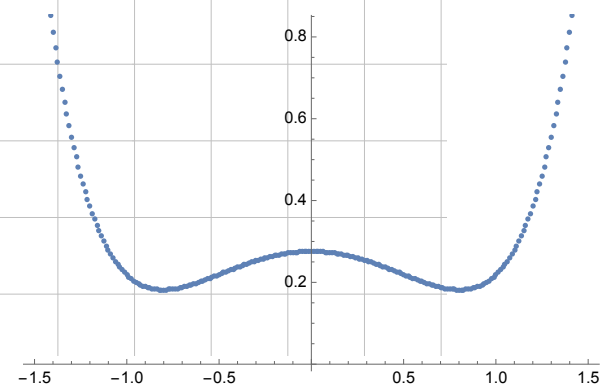


$$\partial_t U_k(\tilde{\xi}) = -d U_k(\tilde{\xi}) + (d-2) \tilde{\xi} U_k'(\tilde{\xi}) + v_d \int_0^\infty dy y^{\frac{d-2}{2}} \frac{2r(y) - 2yr'(y)}{y + u_k(\tilde{\xi}) + 2\tilde{\xi} u_k''(\tilde{\xi}) + r(y)}$$

$$\text{RG FIXED POINT} \longrightarrow \partial_t U_k^*(\tilde{\xi}) = 0 \quad (\text{ODE})$$

$d=3$ TWO NON-SINGULAR, GLOBALLY DEFINED SOLUTIONS:

- $U_k^*(\tilde{\xi}) = \text{CONST}$ (GAUSSIAN)
- THE WILSON-FISHER F.P.



7 DE(0) (LPA) — RG EIGENVALUES

$$u_k(\phi) = u_k^*(\phi) + \varepsilon \delta u(\phi) e^{\lambda t} \quad \left(\tilde{g} = \frac{1}{2} \phi^2, \phi > 0 \right)$$

Plug into $\partial_t u_k(\phi) = \dots$, expand up to order ε

$$-(\lambda + d) \delta u = \underbrace{\frac{2-d}{2} \phi \delta u'}_{=: \tau(\phi)} + \underbrace{v_d \int_0^\infty dy y^{\frac{d}{2}-1} \frac{-2y^2 r'(y)}{(y + yr(y) + u^*(\phi)')^2}}_{=: \sigma(\phi)} \delta u''$$

$$\left[\sigma(\phi) \partial_\phi^2 + \tau(\phi) \partial_\phi \right] \delta u = -(\lambda + d) \delta u \quad \text{(STURM-LIOUVILLE PROBLEM)}$$

BOUNDARY CONDITIONS: $\delta u'(0) = 0$ (EVEN EIGENPERTURBATIONS)
OR
 $\delta u(0) = 0$ (ODD EIGENPERTURBATIONS)

$$\text{AT } \phi \rightarrow \infty \quad \delta u \simeq C \phi^{\frac{2(d+\lambda)}{d-2}} (1 + \dots) + D e^{\omega(\phi)} \quad \omega(\phi) \sim \phi^{\frac{2d+4}{d-2}}$$

DEMANDING THAT $D = 0 \longrightarrow$ DISCRETE SPECTRUM

$$\underbrace{\left[\sigma(\phi) \partial_\phi^2 + \tau(\phi) \partial_\phi \right]}_{=: \mathcal{D}} \delta u = -(\lambda + d) \delta u$$

$$\xi(\phi) \text{ FULFILLS: } \sigma(\phi) \xi'(\phi) = [\tau(\phi) - \sigma'(\phi)] \xi(\phi)$$

$\mathcal{D}$ IS HERMITIAN IN THE SENSE OF $L^2([0, \infty[; \xi)$

$$(g | \mathcal{D} f) = (\mathcal{D} g | f) \quad \left. \begin{array}{l} \} f'(0) = g'(0) = 0 \\ \text{OR} \\ \} f(0) = g(0) = 0 \end{array} \right\}$$

SPECTRUM OF $\mathcal{D}$ IS DISCRETE AND REAL

In[58]:= ComplexListPlot[EigenvaluesLPAList, PlotRange → {-2, 50}]



EigenvaluesLPAList

```
{-1.51419, 0.628622, 1.81255, 3.04806, 4.32256, 5.6306, 6.96687, 8.32779, 9.71032, 11.1121, 12.5313, 13.9668, 15.419, 16.8913, 18.3923,
19.9378, 21.5485, 23.246, 25.0473, 26.9634, 28.9996, 31.1575, 33.4371, 35.8373, 38.3569, 40.9945, 43.7489, 46.6187, 49.6028,
52.7, 55.909, 59.2285, 62.6575, 66.1946, 69.8384, 73.5876, 77.4407, 81.3962, 85.4525, 89.6078, 93.8603, 98.208, 102.649,
107.181, 111.801, 116.507, 121.297, 126.167, 131.114, 136.135, 141.226, 146.382, 151.601, 156.876, 162.203, 167.577, 172.991,
178.439, 183.915, 189.412, 194.926, 200.451, 205.985, 211.531, 217.093, 222.684, 228.316, 234.009, 239.78, 245.645, 251.617,
257.707, 263.922, 270.269, 276.751, 283.374, 290.14, 297.054, 304.12, 311.339, 318.717, 326.258, 333.964, 341.84, 349.89,
358.118, 366.528, 375.125, 383.911, 392.893, 402.074, 411.459, 421.052, 430.858, 440.883, 451.13, 461.605, 472.313, 483.259,
494.448, 505.886, 517.578, 529.53, 541.747, 554.235, 567., 580.047, 593.384, 607.016, 620.949, 635.191, 649.746, 664.623,
679.828, 695.367, 711.248, 727.478, 744.065, 761.015, 778.336, 796.036, 814.123, 832.604, 851.489, 870.784, 890.499, 910.641,
931.221, 952.246, 973.725, 995.669, 1018.08, 1040.98, 1064.37, 1088.27, 1112.67, 1137.6, 1163.06, 1189.06, 1215.62, 1242.74,
1270.44, 1298.73, 1327.61, 1357.11, 1387.23, 1417.98, 1449.38, 1481.44, 1514.17, 1547.59, 1581.7, 1616.53, 1652.08, 1688.37,
```

8 DE(2) - RG EIGENVALUES

$$|\bar{V}(\phi)\rangle = \begin{pmatrix} v(\phi) \\ z(\phi) \end{pmatrix}$$

$$\text{THE EIGENPROBLEM: } \mathbb{D} |\bar{V}(\phi)\rangle = \lambda |\bar{V}(\phi)\rangle$$

$$\mathbb{D} = \mathbb{K}(\phi) \partial_\phi^2 + \mathbb{T}(\phi) \partial_\phi + \mathbb{A}(\phi)$$

$$\mathbb{K}(\phi) = \begin{pmatrix} \Sigma_{vv}(\phi) & 0 \\ \Sigma_{zv}(\phi) & \Sigma_{zz}(\phi) \end{pmatrix}$$

$$\mathbb{T}(\phi) = \begin{pmatrix} T_{vv}(\phi) & T_{vz}(\phi) \\ T_{zv}(\phi) & T_{zz}(\phi) \end{pmatrix}$$

$$\mathbb{A}(\phi) = \begin{pmatrix} \Lambda_{vv}(\phi) & \Lambda_{vz}(\phi) \\ 0 & \Lambda_{zz}(\phi) \end{pmatrix}$$

Follow from
F-P solution

No obvious hermiticity of $\mathbb{D}$.

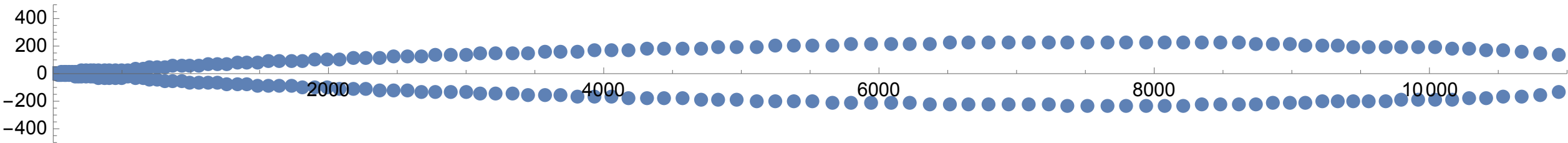
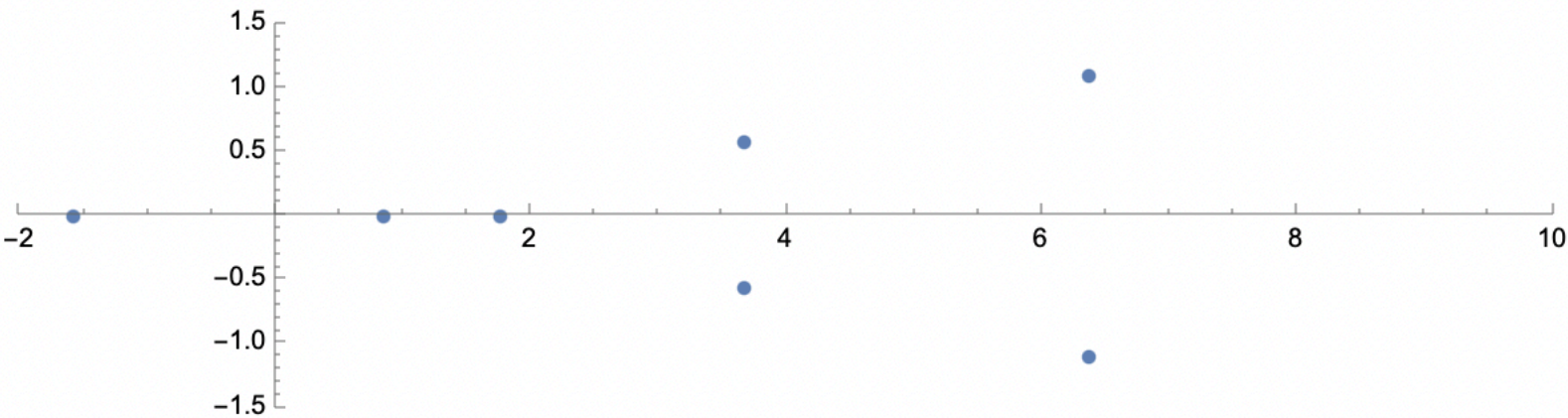
```
Sort[Eigenvalues[calculatedMatrix]]
```

```
{-1.59163 + 0. i, 0.852621 + 0. i, 1.76576 + 0. i, 3.66089 - 0.569525 i, 3.66089 + 0.569525 i, 6.37467 - 1.1005 i, 6.37467 + 1.1005 i, 10.1109 - 1.51881 i, 10.1109 + 1.51881 i, 15.6977 - 0.640652 i, 15.6977 + 0.640652 i, 19.355 + 0. i, 25.912 - 2.66709 i, 25.912 + 2.66709 i, 35.7314 - 4.35271 i, 35.7314 + 4.35271 i, 47.3972 - 5.90052 i, 47.3972 + 5.90052 i, 60.9297 - 7.4482 i, 60.9297 + 7.4482 i, 76.3324 - 9.03234 i, 76.3324 + 9.03234 i, 93.6059 - 10.6672 i, 93.6059 + 10.6672 i, 112.75 - 12.3592 i,
```

```
Sort[Eigenvalues[calculatedMatrix]]
```

```
{-1.59163 + 0. i, 0.852621 + 0. i, 1.76576 + 0. i, 3.66089 - 0.569525 i, 3.66089 + 0.569525 i, 6.37467 - 1.1005 i, 6.37467 + 1.1005 i, 10.1109 - 1.51881 i, 10.1109 + 1.51881 i, 15.6977 - 0.640652 i, 15.6977 + 0.640652 i, 19.355 + 0. i, 25.912 - 2.66709 i, 25.912 + 2.66709 i, 35.7314 - 4.35271 i, 35.7314 + 4.35271 i, 47.3972 - 5.90052 i, 47.3972 + 5.90052 i, 60.9297 - 7.4482 i, 60.9297 + 7.4482 i, 76.3324 - 9.03234 i, 76.3324 + 9.03234 i, 93.6059 - 10.6672 i, 93.6059 + 10.6672 i, 112.75 - 12.3592 i,
```

```
ComplexListPlot[Eigenvalues[calculatedMatrix], PlotRange -> {{-2, 10}, {-1.5, 1.5}}]
```



CFT:

$$\omega = 0.8297(2)$$
$$\omega_2 = 1.67 - 2.02$$
$$\omega_3 = 3.8 - 4.2$$
$$\omega_4 = 5.5 - 6.2$$
$$\omega_5 = 7.2 - 8.0$$

Going to higher order in DE does not help.

Can the complex eigenvalues represent the actual physical truth?

How to organise the calculation so that the spectrum remains real?

