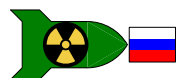


Monogamy and polygamy of Bell inequalities

Adam Bednorz

2026-10-16



Bell, CHSH and monogamy

Bell (1964), Clauser, Horne, Shimony, Holt (1969)

$$S_{AB} = A_1(B_1 + B_2) + A_2(B_1 - B_2), \quad S_{AB} = \langle S_{AB} \rangle.$$

Local hidden variables

For $b_1, b_2 = \pm 1$, one bracket vanishes and the other equals ± 2 .

$$|S_{AB}| \leq 2$$

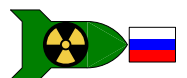
Quantum theory

For $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$,

$$A_1 = Z, \quad A_2 = X,$$

$$\sqrt{2}B_{1,2} = Z \pm X$$

$$S_{AB} = 2\sqrt{2}$$



Cirel'son's bound (1980)

For Hermitian involutions on separate parties,

$$\mathcal{S}_{AB}^2 = 4 - [A_1, A_2][B_1, B_2].$$

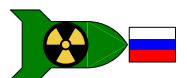
$$\|[A_1, A_2]\| \leq 2, \|[B_1, B_2]\| \leq 2,$$

since $|\psi_j\rangle = A_j|\psi\rangle$, $|\langle[A_1, A_2]\rangle| = 2\text{Im}\langle\psi_1|\psi_2\rangle$

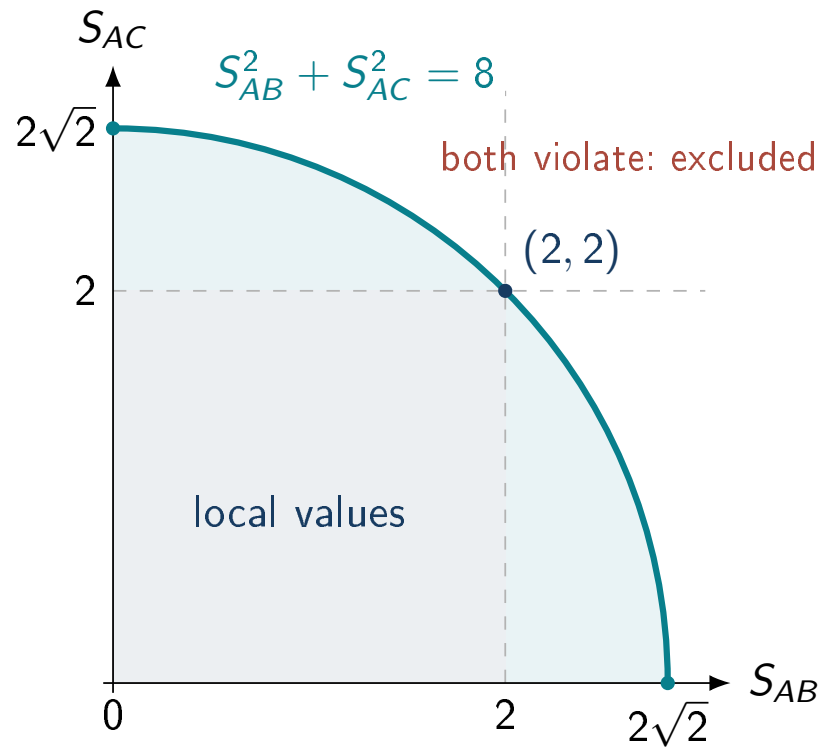
$$\Rightarrow \|\mathcal{S}_{AB}\|^2 \leq 8.$$

Dimension-independent conclusion

$$|\langle\mathcal{S}_{AB}\rangle| \leq \|\mathcal{S}_{AB}\| \leq 2\sqrt{2}.$$



Toner-Verstraete (2006): Bell monogamy



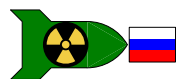
Shared Alice settings

$$S_{AB}^2 + S_{AC}^2 \leq 8$$

The same A_1, A_2 occur in both tests.

If $|S_{AB}| > 2$, then $|S_{AC}| < 2$.

At $|S_{AB}| = 2\sqrt{2}$, $S_{AC} = 0$.



Every point of the TV boundary is attained

$$\sqrt{2} |\psi_t\rangle = |100\rangle + \cos t |010\rangle + \sin t |001\rangle, \quad 0 \leq t \leq \frac{\pi}{2}.$$

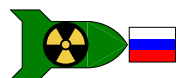
Choose *fixed* measurements for the entire family:

$$A_1 = X, \quad A_2 = Y, \quad B_{1,2} = C_{1,2} = \frac{X \pm Y}{\sqrt{2}}.$$

Direct substitution

$$(S_{AB}, S_{AC}) = 2\sqrt{2} (\cos t, \sin t).$$

$t = 0$: Bell pair AB plus spectator C . $t = \pi/4$: $(2, 2)$.



From CHSH to four Pauli correlations

In each Jordan block, choose common local X, Z axes:

$$D_1 = c_D Z_D + s_D X_D, \quad D_2 = c_D Z_D - s_D X_D, \quad c_D^2 + s_D^2 = 1.$$

Then $\mathcal{S}_{AB}/2$ has coefficients

$$(c_A c_B, s_A c_B, c_A s_B, -s_A s_B)$$

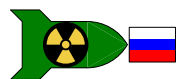
against (ZZ, XZ, ZX, XX) . Their squared norm equals 1.

Cauchy–Schwarz

$$\frac{S_{AB}^2}{4} \leq Q_{AB} := \langle X_A X_B \rangle^2 + \langle X_A Z_B \rangle^2 + \langle Z_A X_B \rangle^2 + \langle Z_A Z_B \rangle^2.$$



The Alice axes are the same in Q_{AB} and Q_{AC} .



The anticommuting uncertainty lemma

Let $F_i = F_i^\dagger$, $F_i^2 = I$, and $\{F_i, F_j\} = 0$ for $i \neq j$.

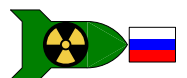
$$f_i = \langle F_i \rangle, \quad R = \sum_i f_i^2, \quad H = \sum_i f_i F_i.$$

The anticommutators cancel every cross term:

$$H^2 = R, \quad \langle H \rangle = R.$$

Positivity of the variance

$$0 \leq \langle H^2 \rangle - \langle H \rangle^2 = R - R^2 \quad \Rightarrow \quad \boxed{\sum_i \langle F_i \rangle^2 \leq 1}.$$



A two-clique proof of TV monogamy

Write Pauli strings in the order ABC ; I means an inactive party.

Clique 1: pairwise
anticommuting

Clique 2: pairwise
anticommuting

ZZI, ZXI, XIZ, XIX

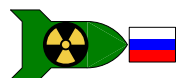
XZI, XXI, ZIZ, ZIX

$$\sum_{F \in C_1} \langle F \rangle^2 \leq 1$$

$$\sum_{F \in C_2} \langle F \rangle^2 \leq 1$$

These eight strings exhaust $Q_{AB} + Q_{AC}$. Hence

$$\frac{S_{AB}^2 + S_{AC}^2}{4} \leq Q_{AB} + Q_{AC} \leq 2.$$



Mermin (1990): four triples compete on four parties

$$\begin{aligned}\mathcal{M}_{ABC} = & A_1 B_1 C_2 + A_1 B_2 C_1 \\ & + A_2 B_1 C_1 - A_2 B_2 C_2.\end{aligned}$$

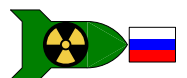
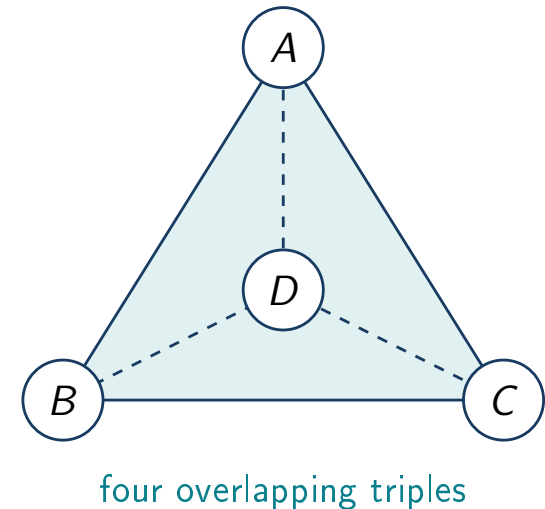
Local bound: $|M_{ABC}| \leq 2$. Quantum
maximum: 4.

Four-party trade-off

$$M_{ABC}^2 + M_{ABD}^2 + M_{ACD}^2 + M_{BCD}^2 \leq 16.$$

Not all four triples can violate $|M| \leq 2$.

This statement does *not* prohibit every pair of simultaneous violations.



One normalization for the MABK family

MABK=Mermin-Ardehali-Belinskii-Klyshko (1990,1992,1993)

Set $D^{(j)} = D_1^{(j)} + iD_2^{(j)}$ and define

$$\mathcal{A}_n = 2^{-(n-1)/2} \operatorname{Re} \left[e^{i(n-1)\pi/4} \prod_{j=1}^n D^{(j)} \right].$$

All local bounds equal one

$$|A_n| \leq 1.$$

Deterministic $D_1 + iD_2$ has modulus $\sqrt{2}$ and an odd multiple of $\pi/4$ as phase.

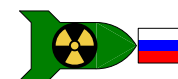
Translation to familiar conventions

Up to outcome/setting relabellings,

$$A_2 = S/2, \quad A_3 = M_3/2, \quad A_4 = M_4/4.$$



Quantum maximum: $|A_n| \leq 2^{(n-1)/2}, \quad n \geq 2.$



The MABK-to-Pauli-squares mapping

$$D_1^{(j)} + iD_2^{(j)} = (1 + i)(c_j Z_j - is_j X_j), \quad c_j^2 + s_j^2 = 1.$$

The product expansion has one coefficient per X/Z string:

$$\mathcal{A}_n = \sum_{\nu \in \{X, Z\}^n} u_\nu P_\nu, \quad u_\nu = \pm \prod_{j: \nu_j = Z} c_j \prod_{j: \nu_j = X} s_j.$$

The phase contributes only a sign. The coefficient vector has unit norm:

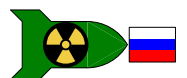
$$\sum_{\nu} u_\nu^2 = \prod_j (c_j^2 + s_j^2) = 1.$$

Cauchy–Schwarz:

$$|A_n|^2 \leq Q_{1\dots n} := \sum_{\nu \in \{X, Z\}^n} \langle P_\nu \rangle^2.$$



Żukowski, Brukner (2002)



Uffink (2002): an independent MABK maximizer

For $n \geq 2$, the Uffink-type quadratic bound implies

$$|\langle \tilde{\mathcal{A}}_n \rangle| = \left| \left\langle \prod_{j=1}^n (D_1^{(j)} + iD_2^{(j)}) \right\rangle \right| \leq 2^{n-1}.$$

Thus $|\tilde{A}_n| \leq 2^{(n-1)/2}$, independently of a clique cover.

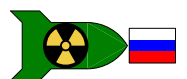
Saturation by a phase-matched GHZ state

$$D_1 = X, \quad D_2 = Y, \quad \prod_j (X_j + iY_j) = 2^n |0^n\rangle\langle 1^n|,$$

$$|\text{GHZ}_\phi\rangle = \frac{|0^n\rangle + e^{i\phi} |1^n\rangle}{\sqrt{2}}, \quad \phi = -\frac{(n-1)\pi}{4}.$$



For $n = 2$: $\langle A_1 B_1 - A_2 B_2 \rangle^2 + \langle A_1 B_2 + A_2 B_1 \rangle^2 \leq 4$.



Two graphs that must not be confused

Physical overlap pattern

Vertices are **parties**.

A tested subset is a hyperedge.

Example: ABC , ABD , ACD , BCD .

Anticommutation graph G

Vertices are **Pauli strings** F_i .

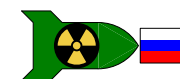
Edges mean $\{F_i, F_j\} = 0$.

Nonedges mean $[F_i, F_j] = 0$.

$$F_i F_j = (-1)^{h(i,j)} F_j F_i,$$

where h counts distinct nonidentity Pauli factors on common active sites.

Kurzyński, Paterek, Ramanathan, Laskowski, Kaszlikowski (2011)



From integer covers to fractional covers

For each anticommuting clique C , assign a weight $w_C \geq 0$ such that

$$\sum_{C \ni i} w_C \geq 1 \quad \text{for every vertex } i.$$

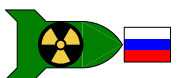
Then

$$\sum_i \langle F_i \rangle^2 \leq \sum_C w_C \sum_{i \in C} \langle F_i \rangle^2 \leq \sum_C w_C.$$

Minimize the price of covering all strings

$$\bar{\chi}_f(G) = \min \left\{ \sum_C w_C : w_C \geq 0, \sum_{C \ni i} w_C \geq 1 \right\}.$$

Integer covers require whole cliques. Fractional covers allow overlap without paying twice.



Lovász theta: the convention and the SDP

$$\alpha(G) \leq \beta(G) \leq \vartheta(G) \leq \bar{\chi}_f(G) \leq \bar{\chi}(G).$$

Primal form

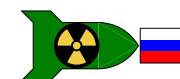
$$\begin{aligned} \vartheta(G) = \max_X \langle J, X \rangle \\ \text{s.t. } X \succeq 0, \quad \text{tr } X = 1, \\ X_{ij} = 0 \quad (ij \in E). \end{aligned}$$

Dual certificate

$$\begin{aligned} \vartheta(G) = \min_{\lambda, \Gamma} \lambda \\ \text{s.t. } \Gamma \succeq 0, \quad \Gamma_{ii} = \lambda - 1, \\ \Gamma_{ij} = -1 \quad (ij \notin E, i \neq j). \end{aligned}$$

J is the all-ones matrix; dual edge entries are free. $\vartheta(K_m) = 1$,
 $\vartheta(\overline{K}_m) = m$.

$\beta(G)$ optimizes over involutions with the full prescribed
(anti)commutation relations. Lovász (1979), Knuth (1994)



Why ϑ bounds squared expectations

Put $f_i = \langle F_i \rangle$, $R = \sum_i f_i^2 > 0$, and define the real Gram matrix

$$K_{ij} = f_i f_j \operatorname{Re} \langle F_i F_j \rangle.$$

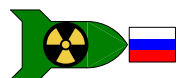
Then $K \succeq 0$, $\operatorname{tr} K = R$, and $K_{ij} = 0$ on anticommuting edges.

$$H = \sum_i f_i F_i, \quad \langle J, K \rangle = \langle H^2 \rangle \geq \langle H \rangle^2 = R^2.$$

$X = K/R$ is a feasible theta matrix

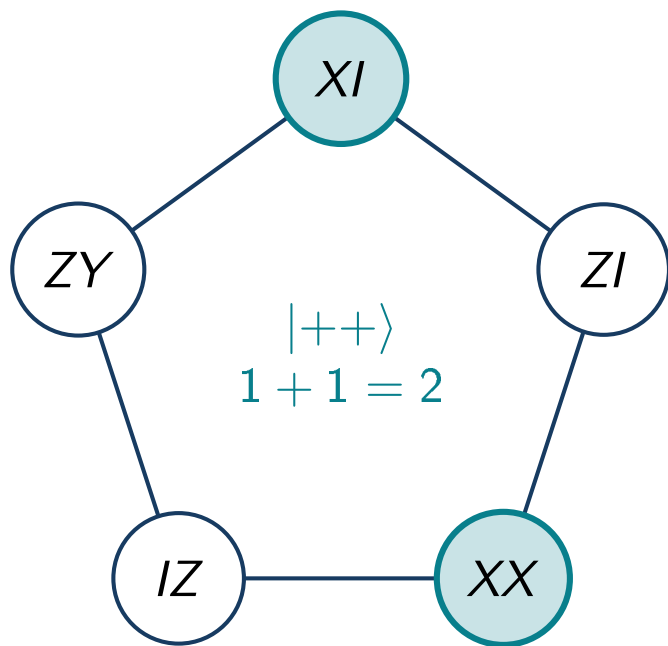
$$\boxed{\vartheta(G) \geq \langle J, K/R \rangle \geq R.}$$

de Gois, Hansenne, Gühne (2023)



A pentagon where ϑ is not tight

	α	β	ϑ	$\bar{\chi}_f$	$\bar{\chi}$
C_5	2	2	$\sqrt{5}$	5/2	3



These two-qubit strings anticommute exactly on the cycle edges.

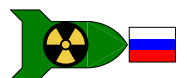
In $|++\rangle$, the highlighted commuting pair has squared expectations $1 + 1 = 2$.

A genuine relaxation gap

$$\max \sum_{i=1}^5 \langle F_i \rangle^2 = 2 < \sqrt{5}.$$



Xu, Schwonnek, Winter (2024)



Odd-cycle proof: contract three vertices into one

Local lemma. For $[A, C] = 0$ and $\{A, B\} = \{B, C\} = 0$,

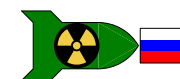
$$\langle A \rangle^2 + \langle B \rangle^2 + \langle C \rangle^2 \leq 1 + \langle AC \rangle^2.$$

On C_m with odd $m \geq 5$, replace (B_1, B_2, B_3) by $D = B_1 B_3$:

$$\sum_{j=1}^m \langle B_j \rangle^2 \leq 1 + \langle D \rangle^2 + \sum_{j=4}^m \langle B_j \rangle^2.$$

The remaining operators form C_{m-2} . Telescope down to C_3 :

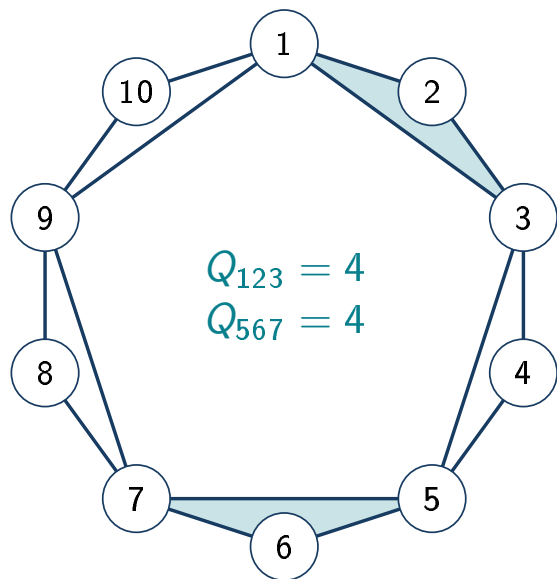
$$\boxed{\beta(C_{2n+1}) = n}, \quad \vartheta(C_m) = \frac{m \cos(\pi/m)}{1 + \cos(\pi/m)} > n \quad (m \geq 5).$$



Back to a physical cycle of overlapping triples

For $m = 2n + 1$, use $2m$ physical parties:

$$T_j = (2j - 1, 2j, 2j + 1) \pmod{2m}.$$



$$\sum_{j=1}^m Q_{T_j} \leq 4n.$$

For five triples on ten parties:

$$8 \text{ (true maximum)} < 4\sqrt{5} \text{ (theta)}.$$

Proof route: four string families $\rightarrow$ pair compression $\rightarrow$ four odd-cycle bounds.

Saturation: GHZ-type states on n disjoint triples, with product spectators.



Tran, Ramanathan, McKague, Kaszlikowski, Paterek (2018)



Three ways to escape the original trade-off

Change the functional

Śliwa inequality 5

Lower-order correlations enter.

Four triples violate together.

Change the system size

MABK on five parties

Five overlapping four-party tests.

All five subsets violate.

Change the setting pattern

Three-setting triangle

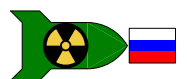
Incoming and outgoing tests use different setting pairs.

All three edges violate.

No contradiction

Each construction changes a hypothesis of a particular monogamy statement.

Cieśliński, Knips, Kowalczyk, Laskowski, Paterek, Vértesi, and Weinfurter (2024)



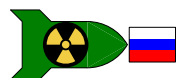
Śliwa (2003): 46 classes of ABC inequalities

$$\begin{aligned} \mathcal{W}_5 = & A_1 + B_1 + C_1 - (A_2 B_2 + B_2 C_2 + C_2 A_2) \\ & + (A_1 B_2 + A_2 B_1 + B_1 C_2 + B_2 C_1 + A_1 C_2 + A_2 C_1) \\ & - A_1 B_1 C_1 + A_2 B_2 C_2 \\ & - (A_2 B_1 C_1 + A_1 B_2 C_1 + A_1 B_1 C_2). \end{aligned}$$

Single three-party test

Local bound: $\mathcal{W}_5 \leq 3$.

Quantum maximum: $8\sqrt{5} - 13$.



Śliwa 5: all four triples of ABCD violate

Use a permutation-symmetric four-qubit state,

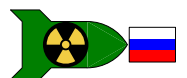
$$|\psi\rangle = \sum_{k=0}^4 \psi_k |D_4^k\rangle, \quad |D_4^k\rangle = \binom{4}{k}^{-1/2} \sum_{|x|=k} |x\rangle,$$

$$(\psi_0, \dots, \psi_4) \simeq (0.455452, 0.720303, 0.315565, -0.362381, 0.206945).$$

At every party: $D_1 = Z$, $D_2 = Z \cos \phi + X \sin \phi$, $\phi \simeq 1.975238$.

Identical three-party marginals

$$W_{ABC} = W_{ABD} = W_{ACD} = W_{BCD} \simeq 3.220475 > 3.$$



Five parties: every four-party MABK test violates

For a four-party subset T , use the unnormalized convention

$$\mathcal{M}_T = \text{Re} \left[(1 - i) \prod_{j \in T} (X_j + iY_j) \right], \quad |M_T| \leq 4 \text{ locally.}$$

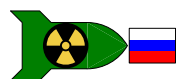
$$|\psi_5\rangle = \frac{|11111\rangle}{\sqrt{2}} + \frac{e^{-i\pi/4}}{\sqrt{10}} \sum_{j=1}^5 |1_j 0_{\neq j}\rangle.$$

All five leave-one-out tests have the same value

$$M_T = 8\sqrt{\frac{2}{5}} \simeq 5.059644 > 4 \quad \text{for every } |T| = 4.$$



$N = 5$ is the **total** number of parties; each Bell test has four parties.



Triangle CHSH: reuse one setting, not both

$$S_{AB} = \langle (A_1 + A_2)B_1 + (A_1 - A_2)B_3 \rangle,$$

$$S_{BC} = \langle (B_1 + B_2)C_1 + (B_1 - B_2)C_3 \rangle,$$

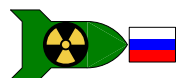
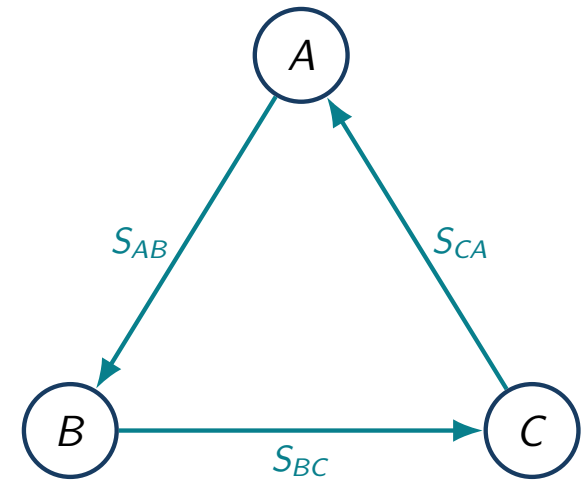
$$S_{CA} = \langle (C_1 + C_2)A_1 + (C_1 - C_2)A_3 \rangle.$$

Each local bound is 2.

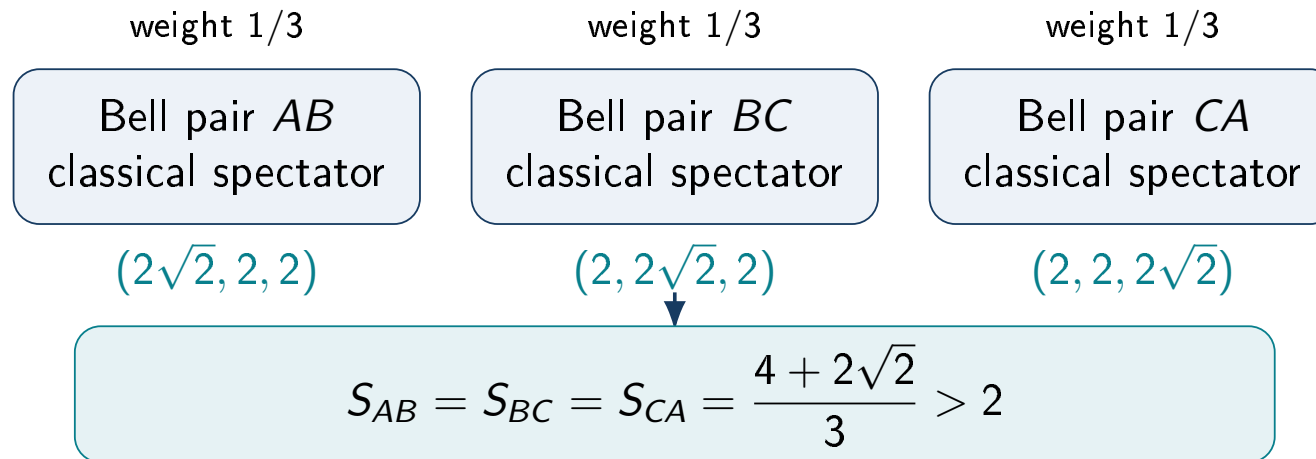
Each party uses settings $\{1, 2\}$ on its outgoing edge, but $\{1, 3\}$ on its incoming edge.

The TV hypothesis has changed

There is no single shared pair of settings at a vertex.



A triangle construction with a simple resource story

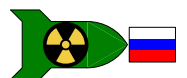


A local orthogonal flag selects one Bell-pair sector. Local dimension $2 + 2 + 1 = 5$ suffices.

Matching quantum upper bound (NPA / sum of squares)

$$|S_{AB} + S_{BC} + S_{CA}| \leq 4 + 2\sqrt{2}.$$

NPA= Navascués, Pironio, Acín (2007)



Semidefinite programming

Constructive lower bound

Optimize a state and local measurements, often by see-saw iteration.

A feasible strategy proves a value *can be reached*.

It does not prove global optimality.

Practical route: exploit symmetries $\rightarrow$ block-diagonal SDP $\rightarrow$ reconstruct and verify a certificate.

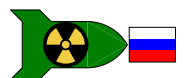
NPA upper bound / SOS certificate

Use moments $\Gamma_{u,v} = \langle u^\dagger v \rangle$ for words in the observables.

Impose $\Gamma \succeq 0$, $D_i^2 = 1$ and cross-party commutation.

A verified identity

$b - \mathcal{B} = \sum_k P_k^\dagger P_k$ proves $B \leq b$.



I_{3322} – three settings two outcomes

$$\begin{aligned} \mathcal{I}(A, B) = & A_1 + A_2 + B_1 + B_2 + (A_1 + A_2)(B_1 + B_2) \\ & + A_3(B_1 - B_2) + (A_1 - A_2)B_3. \end{aligned}$$

Positive-violation convention

$$I \geq -4 \quad \text{locally,}$$

$$I \geq -5 \quad \text{for two qubits.}$$

Thus $I < -5$ requires going beyond a two-qubit realization.

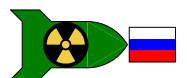
Higher-dimensional behavior

Constructions approach

$$I \simeq -5.0035.$$

The exact unrestricted optimum and the necessity of infinite dimension are not established by those numerics.

Froissart (1981), Śliwa (2003), Collins, Gisin (2004), Pál, Vértesi (2010)



Same sign: AB and AC can both violate

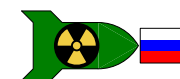
Optimize $I(A, B) + I(A, C)$, local sum ≥ -8 .

Local dimensions	joint value	symmetric value per edge
$2 \times 2 \times 2$	$-8.128866 \dots$	$-4.064433 \dots$
$3 \times 2 \times 2$	$-8.200346 \dots$	$-4.100173 \dots$

Status of the stronger candidate

An explicit qutrit–qubit–qubit strategy reaches the displayed value.

Numerical NPA bounds agree to reported precision; this is *not* an exact global-optimality certificate.



Opposite output signs: a different sharing problem

Now flip *all three Alice outputs in one test*:

$$I(A, B) + I(-A, C) \geq -6 - 2\sqrt{2} \simeq -8.828427.$$

Do not replace this by a scalar subtraction

$$I(-A, C) \neq -I(A, C)$$

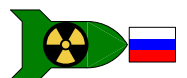
The terms containing only C do not change sign.

A Bell pair AC and classical B attain

$$(I_{AB}, I_{-A,C}) = (-4, -2 - 2\sqrt{2}).$$

Swap B, C and relabel $A \mapsto -A$, then mix with local flags:

$$I_{AB} = I_{-A,C} = -3 - \sqrt{2} \simeq -4.414214 < -4.$$



Cyclic symmetry restores a monogamy statement

For a *common* sign pattern $\sigma = (x, y, z) \in \{\pm 1\}^3$, put

$$I^\sigma(A, B) = I(A, (xB_1, yB_2, zB_3)).$$

Cyclic three-edge bound

$$I^\sigma(A, B) + I^\sigma(B, C) + I^\sigma(C, A) \geq -12.$$

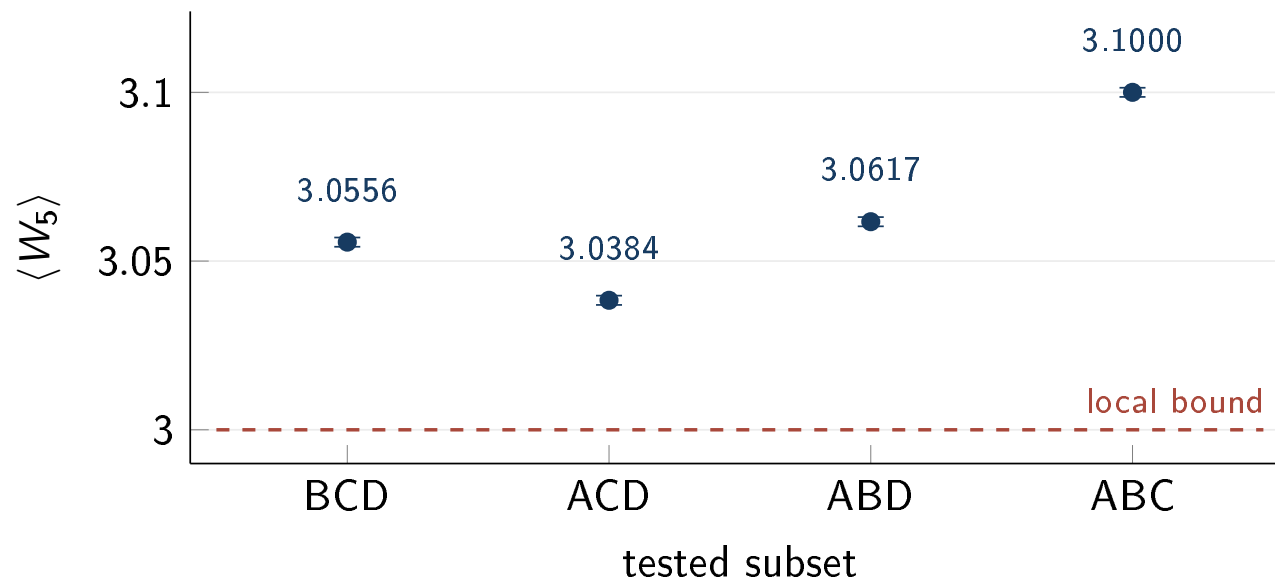
Hence not all three cyclic tests can have $I^\sigma < -4$.

The statement covers all eight *shared* sign patterns.

It is not a theorem about arbitrary independently relabelled inequalities on each edge.



IBM Quantum (kingston) - Śliwa



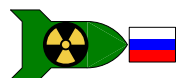
4 physical
qubits

A, B, C, D :
8, 9, 10, 11

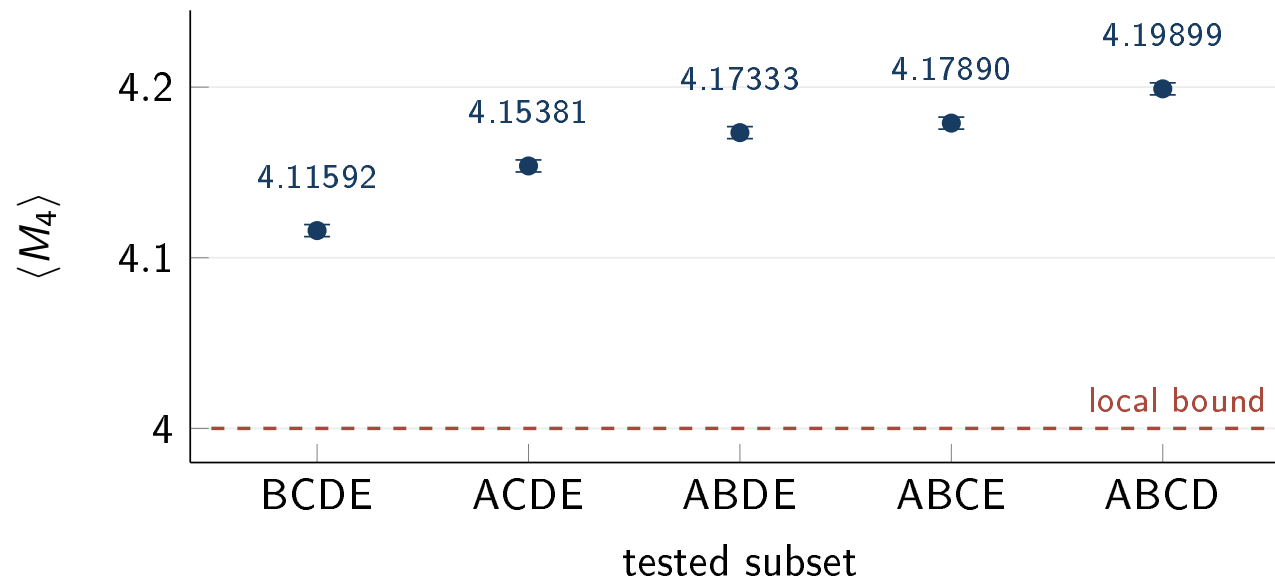
Ideal per
triple
3.220475

Weakest
measured
3.03838
 ± 0.00138

10 jobs $\times$ 12 repetitions $\times$ 10,000 shots per full setting. Bars: 1
shot-noise s.e.



IBM: five four-party MABK violations



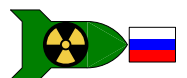
5 physical qubits

122, 123, 136, 143, 144

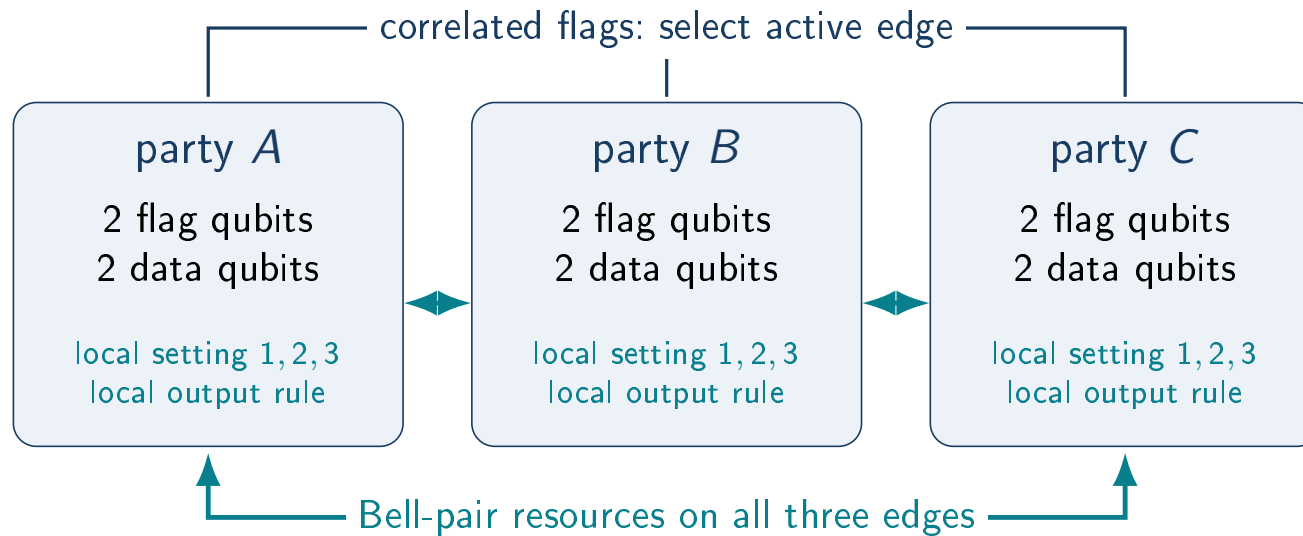
Ideal per subset
5.059644

Weakest measured
4.11592
 ± 0.00353

10 jobs $\times$ 6 repetitions $\times$ 10,000 shots per full setting. Bars: 1 shot-noise s.e.

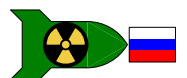


The triangle uses three logical parties, twelve qubits

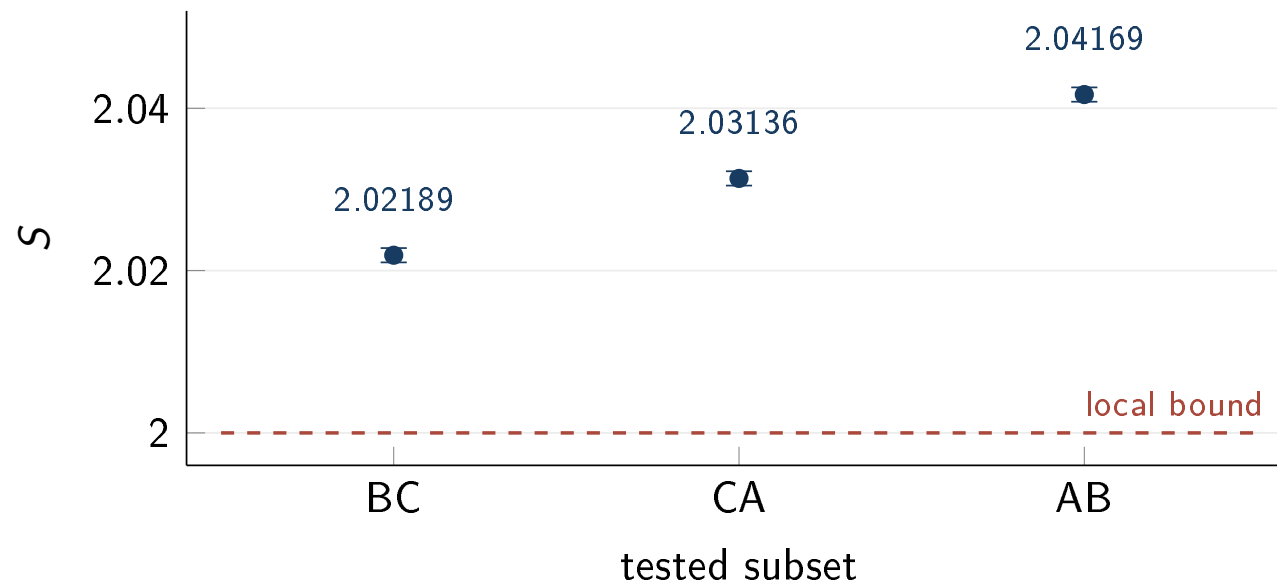


The local flag selects an active Bell-pair outcome or a deterministic output.

All observed flag outcomes are assigned outputs; none are discarded.



IBM: all three triangle CHSH tests exceed two



3 settings per party

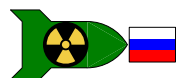
27 full setting combinations

Ideal per edge
2.276142

Weakest measured

2.02189
 ± 0.00088

20 jobs $\times$ 6 repetitions $\times$ 10,000 shots per full setting. Bars: 1 shot-noise s.e.



Perspectives: a sharing profile as a diagnostic

<https://arxiv.org/abs/2609.24835> (Josep Batle, Tomek Rybotycki, Tomek Białeccki, Piotr Garwon, AB)

Benchmark more than one best Bell value

Measure the entire vector of overlapping-subset violations, together with its covariance and its distance to a certified trade-off boundary.

Asymmetry

Compare nominally equivalent subsets: calibration, readout and connectivity sensitivity.

Stability

Repeat in time; separate finite-shot uncertainty from drift and systematic shifts.

Consistency

Audit shared settings, crosstalk and leakage before interpreting a boundary crossing.

