Exercises
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1 Exercises to Methods of Hilbert spaces

1.1 Banach spaces

Hölder inequality:

$$\|fg\|_p \leq \|f\|_q \|g\|_r, \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{r}, \quad 1 \leq p, q, r \leq \infty. \tag{1.1}$$

Minkowski inequality

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p, \quad 1 \leq p \leq \infty. \tag{1.2}$$

Zadanie 1 Jeśli $p \leq r$, $x \in \mathbb{C}^n$, to

$$\|x\|_p \leq n^{\frac{1}{p} - \frac{1}{r}} \|x\|_r.$$

Zadanie 2 Jeśli $1 \leq q \leq r \leq \infty$, $x \in l^q$, to $\|x\|_q \geq \|x\|_r$. Zatem $l^q \subset l^r$.

Solution. Najpierw pokażemy, że $1 \leq p$, to

$$\|x\|_p \leq \|x\|_1. \tag{1.3}$$

Niech $y_i = [0, \dots, x_i, \dots 0]$. Wtedy $x = y_1 + \cdots + y_n$. Stosujemy nierówność Minkowskiego

$$\|x\|_p \leq \|y_1\|_p + \cdots + \|y_n\|_p = |x_1| + \cdots |x_n| = \|x\|_1.$$

Następnie stosujemy (1.3) do $y = [x_1^q, \cdots, x_n^q]$:

$$\Big(\sum |x_i|^r \Big)^{\frac{q}{r}} = \|y\|_{\frac{r}{q}} \leq \|y\|_1 = \sum |x_i|^q.$$

Zadanie 3 Jeśli $1 \leq p \leq r \leq \infty$, $f \in L^r[0,1]$, to

$$\|f\|_p \leq \|f\|_r.$$

Zatem $L^p[0,1] \supset L^r[0,1]$.

Zadanie 4 A linear operator from $\mathbb{C}^m$ to $\mathbb{C}^n$ can be defined by a matrix $[a_{ij}]$.

(1) Jeśli $\mathbb{C}^m$ jest wyposażone w normę $\|\cdot\|_1$ a $\mathbb{C}^n$ w normę $\|\cdot\|_\infty$, wtedy $\|A\| = \max\{|a_{ij}|\}$.
(2) Jeśli $\mathbb{C}^m$ jest wyposażone w normę $\|\cdot\|_\infty$ a $\mathbb{C}^n$ w normę $\|\cdot\|_1$, wtedy $\|A\| \leq \sum_{i,j} |a_{ij}|$.
(3) Jeśli $\mathbb{C}^m$ jest wyposażone w normę $\|\cdot\|_1$ a $\mathbb{C}^n$ w normę $\|\cdot\|_1$, wtedy $\|A\| = \max_j\{\sum_i |a_{ij}|\}$.
(4) Jeśli $\mathbb{C}^m$ jest wyposażone w normę $\|\cdot\|_\infty$ a$\mathbb{C}^n$ w normę $\|\cdot\|_\infty$, wtedy $\|A\| = \max_i\{\sum_j |a_{ij}|\}$.

1.2 Fourier series

Zadanie 5 Consider $\mathbb{C}^n$ with the canonical basis $(\delta_j \ : \ j = 0, 1, \ldots, n-1)$. Define the operators

$$U := \sum_{j=0}^{n-2} |\delta_{j-1})(\delta_j| + |\delta_{n-1})(\delta_0|, \quad R = |\delta_0)(\delta_0| + \sum_{j=1}^{n-1} |\delta_{n-j})(\delta_j|.$$

(i) Show that U and R are unitary.
(ii) Show that $UR = RU^*$ and $(U + U^*)R = R(U + U^*)$.
(iii) Find an orthonormal basis that diagonalizes U.
(iv) Find an orthonormal basis that diagonalizes $U + U^*$ and R.

Solution. (i) is obvious, because both U and R permute an orthonormal basis. The basis

$$e_j = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \mathrm{e}^{\frac{\mathrm{i}jk2\pi}{n}} \delta_k$$

diagonalizes U:

$$Ue_j = \mathrm{e}^{\frac{\mathrm{i}j2\pi}{n}} e_j, \quad U = \sum_{j=0}^{n-1} \mathrm{e}^{\frac{\mathrm{i}j2\pi}{n}} |e_j)(e_j|,$$

We have $Re_j = e_{-j}$. The basis

$$e_0^+ = e_0, \quad e_{\frac{n}{2}}^+ = e_{\frac{n}{2}}^+ \text{ if } n \text{ is even} \tag{1.4}$$

$$e_j^+ = \frac{1}{\sqrt{2}}(e_j + e_{-j}), \quad 0 < j < \left[\frac{n}{2}\right], \tag{1.5}$$

$$e_j^- = \frac{1}{\mathrm{i}\sqrt{2}}(e_j - e_{-j}), \quad 0 < j < \left[\frac{n}{2}\right]. \tag{1.6}$$

diagonalizes simultaneously $U + U^*$ and R:

$$(U + U^*)e_j^+ = 2\cos\frac{j2\pi}{n} e_j^+, \qquad Re_j^+ = e_j^+ \tag{1.7}$$

$$(U + U^*)e_j^- = 2\cos\frac{j2\pi}{n} e_j^-, \qquad Re_j^- = -e_j^-. \tag{1.8}$$

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Set $\mathcal{F}e_j = \delta_j$, or

$$\mathcal{F} = \sum |\delta_j)(e_j| = \sum_{j,k=0}^{n-1} \frac{1}{\sqrt{n}} \mathrm{e}^{-\frac{\mathrm{i}jk2\pi}{n}} |\delta_j)(\delta_k|.$$

Then

$$\mathcal{F}U\mathcal{F}^* = \sum_{j=0}^{n-1} \mathrm{e}^{\frac{\mathrm{i}j2\pi}{n}} |\delta_j)(\delta_j|.$$

Zadanie 6 Consider $L^2[-\pi,\pi]$, where $[-\pi,\pi]$ is treated as the circle. Define the operators

$$U(t)f(\phi) := f(\phi - t), \qquad Rf(\phi) := f(-\phi).$$

(i) Show that $U(t)$ and R are unitary.
(ii) Show that $U(t)U(s) = U(t+s)$ and $U(t)R = RU(-t)$.
(iii) Find an orthonormal basis that diagonalizes $U(t)$.
(iv) Find an orthonormal basis that diagonalizes $U(t) + U(-t)$ and R.

Solution. The on. basis

$$e_j(\phi) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}j\phi}$$

diagonalizes $U(t)$:

$$U(t)e_j = \mathrm{e}^{\mathrm{i}jt} e_j, \quad U(t) = \sum_{j=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}jt} |e_j)(e_j|,$$

We have $Re_j = e_{-j}$. The basis

$$e_0^+ = e_0^+, \quad e_0^+(\phi) = \frac{1}{\sqrt{2\pi}}; \tag{1.9}$$

$$e_j^+ = \frac{1}{\sqrt{2}}(e_j + e_{-j}), \quad e_j^+(\phi) = \frac{1}{\sqrt{\pi}} \cos(j\phi), \quad j = 1, \ldots, \tag{1.10}$$

$$e_j^- = \frac{1}{\mathrm{i}\sqrt{2}}(e_j - e_{-j}), \quad e_j^-(\phi) = \frac{1}{\sqrt{\pi}} \sin(j\phi), \quad j = 0, 1, 2, \ldots. \tag{1.11}$$

diagonalizes simultaneously $U(t) + U(-t)$ and R:

$$(U(t) + U(-t))e_j^+ = 2\cos(jt)e_j^+, \qquad Re_j^+ = e_j^+ \tag{1.12}$$

$$(U(t) + U(-t))e_j^- = 2\cos(jt)e_j^-, \qquad Re_j = -e_j^-. \tag{1.13}$$

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Let $\{\delta_j \ : \ j \in \mathbb{Z}\}$ denote the canonical basis in $l^2(\mathbb{Z})$. Define the unitary Fourier transformation $\mathcal{F} : L^2[-\pi,\pi] \to l^2(\mathbb{Z})$ as

$$\mathcal{F} = \sum_{j=-\infty}^{\infty} |\delta_j)(e_j|,$$

or

$$(\mathcal{F}f)_j = \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{-\mathrm{i}j\phi} f(\phi) \mathrm{d}\phi.$$

The Fourier transformation diagonalizes translations:

$$\mathcal{F}U(t)\mathcal{F}^* = \sum_{j=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}jt} |\delta_j)(\delta_j|.$$

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Zadanie 7 Define $L^2_\pm[-\pi,\pi] := \{f \in L^2[-\pi,\pi] \mid f(\phi) = \pm f(-\phi)$. Then $L^2[-\pi,\pi] = L^2_+[-\pi,\pi] \oplus L^2_-[-\pi,\pi]$. Besides, e^+_n, $n = 0,1,2,\ldots$ is an orthonormal basis of $L^2_+[-\pi,\pi]$ and e^-_n, $n = 1,2,\ldots$ of $L^2_-[-\pi,\pi]$.

Zadanie 8 Prove that $\sqrt{\frac{2}{\pi}}\cos n\phi$ $n = 1,2,\ldots$, $\frac{1}{\sqrt{\pi}}$, is an orthogonal basis of $L^2([0,\pi])$.

Prove that $\sqrt{\frac{2}{\pi}}\sin n\phi$, $n = 1,2,\ldots$, is an orthogonal basis of $L^2([0,\pi])$.

Solution Note that

$$L^2_\pm[-\pi,\pi] \ni f \mapsto U_\pm f := \sqrt{2} f\Big|_{[0,\pi]} \in L^2[0,\pi]$$

is a unitary operator and

$$\begin{aligned} U_+ e^+_0 &= \frac{1}{\sqrt{\pi}}, \\ U_+ e^+_n &= \sqrt{\frac{2}{\pi}}\cos(n\phi), \\ U_- e^-_n &= \sqrt{\frac{2}{\pi}}\sin(n\phi). \end{aligned}$$

Niech I będzie zbiorem. Definiujemy

$$l^2(I) := \{(f_i)_{i\in I} \ : \ \sum |f_i|^2 =: \|f\|^2 < \infty\}.$$

Jeśli $\mathcal{H}$ jest przestrzenią Hilberta z bazą ortonormalną $\{e_i \ : \ i \in I\}$, to

$$(\mathcal{F}f)_i := (e_i|f), \quad f \in \mathcal{H}$$

definiuje operator unitarny $\mathcal{F} : \mathcal{H} \to l^2(I)$. Na przykład, transformata Fouriera

$$L^2[-\pi,\pi] \ni f \mapsto \frac{1}{\sqrt{2\pi}}\hat{f} \in l^2(\mathbb{Z})$$

jest takim operatorem, gdzie

$$\hat{f}_n := \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}n\phi} f(\phi)\mathrm{d}\phi.$$

We will write $c_n = \cos(n\phi)$, $s_n = \sin(n\phi)$.

Zadanie 9 Jedne funkcje lepiej jest rozwijać w szereg kosinusów a inne w szereg sinusów:

$$\begin{aligned} 1 &= c_0 \\ &= \frac{1}{\pi}\sum_{m=0}^{\infty} \frac{2}{2m+1} s_{2m+1}, \\ \sin\phi &= s_1 \\ &= \frac{1}{\pi}\sum_{m=1}^{\infty} (\frac{1}{2m-1} - \frac{1}{2m+1}) c_{2m}. \end{aligned}$$

Można wykorzystać

$$\sin\phi\cos(n\phi) = \frac{1}{2}\Big(\sin(n+1)\phi - \sin(n-1)\phi\Big).$$

Zadanie 10 $h(\phi) := (a - \mathrm{e}^{\mathrm{i}\phi})^{-1}$, $a > 1$. Wtedy

$$\hat{h}_n = \begin{cases} 2\pi a^{-n-1}, & n = 0, 1, \ldots; \\ 0, & n = -1, -2, \ldots. \end{cases}$$

Zadanie 11 $h(\phi) := (\mathrm{e}^{\mathrm{i}\phi} - a)^{-1}$, $a < 1$. Wtedy

$$\hat{h}_n = \begin{cases} 0, & n = 0, 1, 2, \ldots; \\ 2\pi a^{-n-1}, & n = -1, -2, \ldots. \end{cases}$$

Zadanie 12 $h(\phi) := \phi$. Wtedy

$$\hat{h}_n = \begin{cases} \frac{\mathrm{i}2\pi(-1)^n}{n}, & n \neq 0 \\ 0. & n = 0. \end{cases}$$

Aby to otrzymać można zauważyć, że możemy napisać

$$\log(1 + \mathrm{e}^{\pm\mathrm{i}\phi}) = \log\left(\mathrm{e}^{\pm\frac{\mathrm{i}}{2}\phi}\cos\frac{\phi}{2}\right) = \pm\mathrm{i}\phi + \log\left(\cos\frac{\phi}{2}\right). \tag{1.14}$$

(Używamy gałęzi głównej logarytmu). Dlatego,

$$h(\phi) = -\mathrm{i}\log(1 + a\mathrm{e}^{\mathrm{i}\phi}) + \mathrm{i}\log(1 + a\mathrm{e}^{-\mathrm{i}\phi}).$$

$$\log(1 + \mathrm{e}^{\pm\mathrm{i}\phi}) = \lim_{a\searrow 0}\log\left(1 + a\mathrm{e}^{\pm\mathrm{i}\phi}\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\mathrm{e}^{\pm\mathrm{i}\phi n}}{n}.$$

Z tego wynika (1.14).

Częściową suma Fouriera

$$h_{(n)}(\phi) := \sum_{|j|\leq n}\frac{\hat{h}_j\mathrm{e}^{\mathrm{i}n\phi}}{2\pi},$$

jest zbieżna punktowo do ϕ na $]-\pi, \pi[$. Ale w otoczeniu $\phi = \pm\pi$ obserwujemy tzw. zjawisko Gibbsa: funkcja $h_{(n)}$ "przestrzeliwuje" wartość funkcji h. Mamy bowiem

$$h_{(n)}(-\pi + \epsilon) = -2\sum_{j=1}^{n}\frac{\sin\epsilon j}{j}.$$

W otoczeniu nieciągłości funkcji h obserwujemy "zafalowanie" funkcji $h_{(n)}$, które w miarę wzrostu n zwęża się, ale nie zmniejsza swej wysokości zachowując swoją wysokość. To zafalowanie ma w granicy ściśle określony kształt (z dokładnością do zwężania), mamy bowiem

$$\lim_{n\to\infty} h_{(n)}\left(-\pi + \frac{y}{n}\right) = -2\int_0^y \frac{\sin x}{x}\mathrm{d}x =: -2F(y).$$

Funkcja F jest nieparzysta, $\lim_{x\to\infty} F(x) = \frac{\pi}{2}$ i ma maksimum dla $y = \pi$ równe

$$G := \int_0^\pi \frac{\sin x}{x} \mathrm{d}x \approx 1,81,$$

zwane stałą Wilbrahama-Gibbsa.

Ta własność sumy częściowej szeregu Fouriera występuje zawsze, kiedy mamy do czynienia z nieciągłą funkcją. Prowadzi ono do tego, że dla funkcji nieciągłej o skoku $a2\pi$ w sumie częściowej szeregu Fouriera będzie skok $4aG > a2\pi$. Mamy $(4G - 2\pi) \approx 0.18$.

Zadanie 13 Rozważmy $l^2(\mathbb{Z})$ z bazą kanoniczną $(\delta_j \ : \ j \in \mathbb{Z})$. Zdefiniujmy operatory

$$U := \sum_{j=-\infty}^{\infty} |\delta_{j+1})(\delta_j|, \quad R = \sum_{j=-\infty}^{\infty} |\delta_{-j})(\delta_j|.$$

(i) Pokazać, że U i R są unitarne.

(ii) Czy istnieje baza ortonormalna w której U jest diagonalny?

(iii) Odwrotna transformata Fouriera $\mathcal{F}^* : l^2(\mathbb{Z}) \to L^2(S^1)$ diagonalizuje U:

$$\frac{1}{2\pi}\mathcal{F}^* U \mathcal{F} = B, \quad (Bf)(\phi) = \mathrm{e}^{\mathrm{i}\phi} f(\phi), \quad f \in L^2(S^1).$$

(iv) Podać operator unitarny $V : l^2(\mathbb{Z}) \to L^2(0,\pi) \oplus L^2(0,\pi)$ taki, że

$$VRV^* = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}, \quad V(U + U^*)V^* = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix},$$

gdzie

$$(Cg)(\phi) = 2\cos\phi g(\phi), \quad g \in L^2(0,\pi).$$

Wskazówka. Najpierw rozwiązać zadanie 6.

Zadanie 14 Pokazać, że jeśli $f^{(n)}$ istnieje, to

$$|\hat{f}_k| \leq |k|^{-n} \int_{-\pi}^{\pi} \big|f^{(n)}(x)\mathrm{d}x\big|.$$

Solution.

$$k^n \hat{f}_k = \int_{-\pi}^{\pi} f(x)\mathrm{i}^n \partial_x^n \mathrm{e}^{\mathrm{i}kx}\mathrm{d}x \tag{1.15}$$

$$= \int_{-\pi}^{\pi} (-\mathrm{i})^n \big(\partial_x^n f(x)\big)\mathrm{e}^{\mathrm{i}kx}.\mathrm{d}x \tag{1.16}$$

Zadanie 15 Pokazać, że jeśli dla $\epsilon > 0$,

$$|\hat{f}_k| \leq \frac{C}{(|k|+1)^{n+1+\epsilon}},$$

to f jest n-krotnie różniczkowalne.

Solution.

$$\partial_x^n f(x) = \frac{1}{2\pi} \sum k^n \hat{f}_k \mathrm{e}^{\mathrm{i}kx}.$$

Zatem

$$|\partial_x^n f(x)| \leq \frac{1}{2\pi} \sum |k^n \hat{f}_k|.$$

1.3 Falki Haara.

Zdefiniujmy

$$\psi_{k,n}(x) := \begin{cases} 2^{k/2}, & 2^{-k}n \leq x < 2^{-k}n + 2^{-k-1}, \\ -2^{k/2}, & 2^{-k}n + 2^{-k-1} \leq x < 2^{-k}(n+1), \\ 0, & x \notin [2^{-k}n, 2^{-k}(n+1)[; \end{cases}$$

$$\phi_{k,n}(x) := \begin{cases} 2^{k/2}, & 2^{-k}n \leq x < 2^{-k}(n+1), \\ 0, & x \notin [2^{-k}n, 2^{-k}(n+1)[. \end{cases}$$

Czasami nazywa się ψ_{00} "falką matką" a ϕ_{00} "falką ojcem".

Zadanie 16 Wprowadźmy operatory unitarne translacji i skalowania

$$\begin{aligned} (U_t f)(x) &:= f(x-t), \\ (W_s f)(x) &:= s^{-\frac{1}{2}} f(s^{-1}x). \end{aligned}$$

Zauważmy, że możemy napisać

$$\psi_{k,n} = W_{2^{-k}} U_n \psi_{00}, \quad \psi_{k,n}(x) = 2^{k/2} \psi_{00}(2^k x - n).$$

Pokazać, że $\{\psi_{k,n} \mid k = 0, 1, 2, \ldots, \quad n = 0, 1, \ldots, 2^k - 1\}$ oraz funkcja ϕ_{00} stanowią bazę ortonormalną $L^2[0,1]$.

Solution. Sprawdzamy najpierw ortonormalność. Oczywiste jest, że $\mathrm{Span}\{\phi_{kn} \mid k \geq 0\}$ jest gęste i zawiera $\{\psi_{k,n} \mid k = 0, 1, 2, \ldots, \quad n = 0, 1, \ldots, 2^k - 1\}$ oraz ϕ_{00}. Przeciwna inkluzja też jest łatwa.

Zadanie 17 (1) Niech $m \in \mathbb{Z}$. Wtedy

$$\mathcal{V}_m := (\mathrm{Span}\{\psi_{k,n} \ : \ k \leq m, \ n =\})^{\mathrm{cl}} = (\mathrm{Span}\{\phi_{k,n} \ : \ k \geq m+1, \ n \in \mathbb{Z}\})^{\mathrm{cl}}. \quad (1.17)$$

(2) $\{\psi_{k,n} \mid k,n \in \mathbb{Z}\}$ stanowią bazę ortonormalną $L^2(\mathbb{R})$.
(3) $\psi_{m,n}$, $n \in \mathbb{Z}$ stanowią bazę ortonormalną w $\mathcal{V}_m \ominus \mathcal{V}_{m+1}$ (w dopełnieniu ortogonalnym do $\mathcal{V}_{m+1}$ wewnątrz $\mathcal{V}_m$).

Solution. (1): $\subset$ jest oczywiste. Mamy

$$\sum_{j=0}^{\infty} 2^{-\frac{j}{2}} \psi_{-j0} = \sqrt{2}\phi_{10}.$$

To pokazuje $\supset$. (2) Najpierw sprawdzamy ortonormalność ψ_{kn}. Oczywiste jest, że $\mathrm{Span}\{\phi_{kn} \mid k,n \in \mathbb{Z}\}$ jest gęste w $L^2(\mathbb{R})$.

1.4 Dystrybucje

Odwzorowania z $\mathcal{D}(\mathbb{R}^d) \to \mathbb{C}$, zwane dystrybucjami, bywają zapisywne w różny sposób, np.:

$$\mathcal{D}(\mathbb{R}^d) \ni \phi \mapsto T(\phi) = \langle T|\phi\rangle = \int T(x)\phi(x)\mathrm{d}x.$$

Spełniają one następujący warunek: dla każdego zwartego $K \subset \mathbb{R}^d$ istnieje N i C takie, że dla $\phi \in \mathcal{D}(\mathbb{R}^d)$ spełniających $\mathrm{supp}\phi \subset K$,

$$|\langle T|\phi\rangle| \leq C \max_{n\leq N} \sup_x |\partial_x^n \phi(x)|.$$

Przykład: jeśli $T \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, to dystrybucją regularną związaną z F, nazywamy

$$\langle T_F|\phi\rangle = \int T(x)f(x)\mathrm{d}x.$$

A oto delta Diraca w $a \in \mathbb{R}^d$:

$$\langle \delta_a|\phi\rangle = \int \delta(x-a)\phi(x)\mathrm{d}x = \phi(a).$$

Zadanie 18 Pokazać, że

$$\mathcal{P}\int \frac{1}{x}\phi(x)\mathrm{d}x := \lim_{\epsilon\searrow 0}\Big(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\Big)\frac{\phi(x)}{x}\mathrm{d}x$$

jest dystrybucją.

Solution. Niech $\mathrm{supp}\phi \subset K$.

$$\begin{aligned}&\mathcal{P}\int \frac{1}{x}\phi(x)\mathrm{d}x\\ =&\Big(\int_{-\infty}^{-1} + \int_{1}^{\infty}\Big)\frac{\phi(x)}{x}\mathrm{d}x + \lim_{\epsilon\searrow 0}\Big(\int_{-1}^{-\epsilon} + \int_{\epsilon}^{1}\Big)\frac{\phi(x)}{x}\mathrm{d}x \;=\; I + II\end{aligned}$$

$$|II| \leq 2\sup|\phi'|, \quad |I| \leq |K|\sup|\phi|.$$

Zadanie 19 Pokazać, że

$$\mathcal{P}\int_0^\infty \frac{1}{x}\phi(x)\mathrm{d}x := \lim_{\epsilon\searrow 0}\Big(\int_\epsilon^\infty \frac{\phi(x)}{x}\mathrm{d}x + \phi(0)\log\epsilon\Big)$$

jest dystrybucją.

Solution.

$$\begin{aligned}&\mathcal{P}\int_0^\infty \frac{1}{x}\phi(x)\mathrm{d}x\\ =&\int_1^\infty \frac{\phi(x)}{x}\mathrm{d}x + \int_0^1 \frac{\phi(x)-\phi(0)}{x}\mathrm{d}x.\end{aligned}$$

Następnie korzystamy z tego, że funkcja

$$x\mapsto\begin{cases}\frac{\phi(x)-\phi(0)}{x}, & x\in]0,1],\\ \phi'(0), & x=0\end{cases}$$

jest ciągła.

Zadanie 20 Zróżniczkować n-krotnie $\frac{1}{2}\theta(x)x^2$

Zadanie 21 Niech δ_a będzie deltą Diraca w punkcie $a\in\mathbb{R}$. Pokazać, że operator $\mathcal{S}(\mathbb{R})\ni f\mapsto T_a f := \delta_a * f\in\mathcal{S}(\mathbb{R})$ rozszerza się do operatora unitarnego na $L^2(\mathbb{R})$. Czy T_a dla $a\to\infty$ jest zbieżny normowo, silnie lub słabo? Ewentualnie policzyć granicę.

1.5 Zbieżność dystrybucji

Mówimy, że ciąg dystrybucji T_n jest zbieżny (w sensie dystrybucyjnym) do dystrybucji T, gdy

$$\langle T_n|\phi\rangle\to\langle T|\phi\rangle,\quad \phi\in\mathcal{D}(\mathbb{R}^d).$$

Zadanie 22 Niech $f\in L^1(\mathbb{R})$, $\int f=1$, $f_\epsilon(x)=\epsilon^{-1}f(x\epsilon^{-1})$. Wtedy

$$\lim_{\epsilon\searrow 0}=\delta.$$

Solution. Niech $\delta>0$.

$$\begin{aligned}\int f_\epsilon(x)\phi(x)\mathrm{d}x-\phi(0) &= \int f_\epsilon(x)\big(\phi(x)-\phi(0)\big)\mathrm{d}x\\ &= \int_{|x|<\delta} f_\epsilon(x)\big(\phi(x)-\phi(0)\big)\mathrm{d}x + \int_{|x|>\delta} f_\epsilon(x)\big(\phi(x)-\phi(0)\big)\mathrm{d}x \quad = \quad I+II;\\ |I| &\le \sup_{|x|<\delta}|\phi(x)-\phi(0)|\int|f(x)|\mathrm{d}x \le \delta\sup_x|\phi'(x)|\int|f(x)|\mathrm{d}x;\\ |II| &\le 2\sup|\phi(x)|\int_{|x|>\delta/\epsilon}|f(x)|\mathrm{d}x.\end{aligned}$$

Ale $\lim_{\epsilon \searrow 0} \int_{|x|>\delta/\epsilon} |f(x)| \mathrm{d}x = 0$. Więc

$$\Big| \int f_\epsilon(x)\phi(x)\mathrm{d}x - \phi(0) \Big| \leq A\delta.$$

Ale $\delta > 0$ było dowolne.

Zadanie 23 Pokazać wzór Sochockiego.

$$\lim_{\epsilon \searrow 0} \frac{1}{x + \mathrm{i}\epsilon} = \mathcal{P}\frac{1}{x} - \mathrm{i}\pi\delta(x).$$

Solution. Mamy

$$\frac{1}{x + \mathrm{i}\epsilon} = \frac{x}{x^2 + \epsilon^2} - \frac{\mathrm{i}\epsilon}{x^2 + \epsilon^2}.$$

Mamy

$$\int \frac{1}{\pi} \int \frac{\epsilon}{x^2 + \epsilon^2} \mathrm{d}x = 1.$$

Więc, z poprzedniego zadania mamy

$$\lim_{\epsilon \searrow 0} \frac{\mathrm{i}\epsilon}{x^2 + \epsilon^2} = \mathrm{i}\pi\delta(x).$$

Podobnie pokazujemy

$$\lim_{\epsilon \searrow 0} \frac{x}{x^2 + \epsilon^2} = \mathcal{P}\frac{1}{x}.$$

1.6 Równania dystrybucyjne

Zadanie 24 Znaleźć wszystkie dystrybucje spełniające

$$k^m T = 0. \tag{1.18}$$

Solution. T musi mieć nośnik $\{0\}$. Zatem musi mieć postać

$$\sum_{j=0}^{n} c_j \delta^{(j)}(k).$$

Wtedy

$$\langle k^m T | \phi \rangle = \sum_{j=m}^{n} c_j (-1)^j \big(k^m \phi(k)\big)^{(j)}\Big|_{k=0} = \sum_{j=m}^{n} c_j (-1)^j j(j-1)\cdots(j-m+1)\phi^{(j-m)}(0).$$

Czyli rozwiązaniem są

$$T = \sum_{j=0}^{m-1} c_j \delta^{(j)}(k).$$

Zadanie 25 Znaleźć wszystkie dystrybucje spełniające

$$kT = 1. \tag{1.19}$$

Solution.

$$T = \mathcal{P}\frac{1}{k} + c\delta(k).$$

Zadanie 26 Znaleźć wszystkie dystrybucje spełniające

$$(k^2 - 1)T = 1. \tag{1.20}$$

Solution.

$$T = \mathcal{P}\frac{1}{k^2 - 1} + c_+\delta(k-1) + c_-\delta(k+1).$$

Zadanie 27 Znaleźć wszystkie dystrybucje spełniające

$$k^2 T = 1. \tag{1.21}$$

Solution. Zdefiniujmy dystrybucję $\mathcal{P}\frac{1}{k^2}$ wzorem

$$\mathcal{P}\int \frac{\phi(k)}{k^2}\mathrm{d}k := \int_{|k|<1} \frac{\phi(k) - \phi(0) - k\phi'(0)}{k^2}\mathrm{d}k + \int_{|k|>1} \frac{\phi(k)}{k^2}\mathrm{d}k.$$

Solutionm jest

$$T = \mathcal{P}\frac{1}{k^2} + c_0\delta(k) + c_1\delta^{(1)}(k).$$

1.7 Transformata Fouriera

Zadanie 28 Niiech $m > 0$. Pokazać, że

$$\int \frac{\mathrm{e}^{-\mathrm{i}\xi s}}{(s + \mathrm{i}m)}\mathrm{d}\ s \quad = -2\pi\mathrm{i}\theta(\xi)\mathrm{e}^{-m\xi}, \tag{1.22}$$

$$\int \frac{\mathrm{e}^{-\mathrm{i}\xi s}}{(s - \mathrm{i}m)}\mathrm{d}\ s \quad = 2\pi\mathrm{i}\theta(-\xi)\mathrm{e}^{-m|\xi|}, \tag{1.23}$$

$$\int \frac{\mathrm{e}^{-\mathrm{i}\xi s}s}{(s^2 + m^2)}\mathrm{d}\ s \quad = \pi\mathrm{i}\mathrm{sgn}(\xi)\mathrm{e}^{-m|\xi|}, \tag{1.24}$$

$$\int \frac{\mathrm{e}^{-\mathrm{i}\xi s}m}{(s^2 + m^2)}\mathrm{d}\ s \qquad = \pi\mathrm{e}^{-m|\xi|}. \tag{1.25}$$

Zadanie 29 Przechodząc do granicy z m do zera w poprzednim zadaniu, pokazać, że

$$\int \theta(\pm x)\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}x = \frac{\mp\mathrm{i}}{k\mp\mathrm{i}0}, \tag{1.26}$$

$$\int \frac{\mathrm{e}^{-\mathrm{i}kx}}{x\pm\mathrm{i}0}\mathrm{d}x = \mp 2\pi\mathrm{i}\theta(\pm x), \tag{1.27}$$

$$\int \mathrm{sgn}(x)\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}x = -2\mathrm{i}\mathcal{P}\Big(\frac{1}{k}\Big), \tag{1.28}$$

$$\int \mathcal{P}\frac{\mathrm{e}^{-\mathrm{i}kx}}{x}\mathrm{d}x = \pi\mathrm{i}\mathrm{sgn}(k), \tag{1.29}$$

Zadanie 30 Pokazać, że dla $\epsilon \geq 0$, $\lambda > -1$, transformata Fouriera funkcji $\theta(x)x^{\lambda}\mathrm{e}^{-\epsilon x}$ jest równa $\mathrm{e}^{-\mathrm{i}(1+\lambda)\frac{\pi}{2}}\Gamma(\lambda+1)(\xi-\mathrm{i}\epsilon)^{-1-\lambda}$.

Zadanie 31 Pokazać, że dla $f \in \mathcal{D}(\mathbb{R})$

$$Tf(x) := \mathcal{P}\int \frac{f(y)}{x-y}\mathrm{d}y$$

należy do $L^2(\mathbb{R})$ i że T rozszerza się do operatora ograniczonego na $L^2(\mathbb{R})$. Policzyć T^2.
Wskazówka. Warto zastosować transformatę Fouriera.

Zadanie 32 Pokazać, że

$$\mathbb{1}_{[-1,1]} * \mathbb{1}_{[-1,1]} = (2-|x|)\mathbb{1}_{[-1,1]}. \tag{1.30}$$

Wiedząc, że transformata Fouriera $\mathbb{1}_{[-1,1]}$ jest równa $\frac{2\sin(\xi)}{\xi}$, policzyć transformatę Fouriera (1.30).

Solution. $\frac{4\sin^2(\xi)}{\xi^2}$.

1.8 Funkcje Greena

Zadanie 33 Znaleźć dystrybucje G spełniające

$$(\partial_x + 1)G(x) = \delta(x). \tag{1.31}$$

Pokazać, że

$$f := G * h$$

spełnia

$$(\partial_x + 1)f(x) = h(x). \tag{1.32}$$

Solution. Metoda 1. Solutionm równania jednorodnego

$$(\partial_x + 1)G_0(x) = 0. \tag{1.33}$$

jest $G_0(x) := c\mathrm{e}^{-x}$. Uzmienniając stałą c dostajemy równanie

$$\partial_x c(x) = \delta(x).$$

Stąd

$$G(x) = c\mathrm{e}^{-x} + \theta(x)\mathrm{e}^{-x}.$$

W szczególności, mamy funkcję Greena retardowaną $G_+(x) := \theta(x)\mathrm{e}^{-x}$ i adwansowaną $G_- := -\theta(-x)\mathrm{e}^{-x}$. Retardowana jest jedyną dystrybucją temperowaną spośród funkcji Greena.

Metoda 2.

$$\hat{G}(k) = \frac{1}{\mathrm{i}k+1}.$$

Więc

$$G(x) = \int \frac{\mathrm{e}^{-\mathrm{i}kx}}{\mathrm{i}k+1}\mathrm{d}k = \theta\mathrm{e}^{-x}.$$

Zadanie 34 Znaleźć dystrybucje G spełniające

$$(\partial_x + x)G(x,y) = \delta(x-y). \tag{1.34}$$

Pokazać, że

$$f(x) := \int G(x,y)h(y)\mathrm{d}x$$

spełnia

$$(\partial_x + x)f(x) = h(x). \tag{1.35}$$

Solution. Solutionm równania jednorodnego

$$(\partial_x + x)G_0(x,y) = 0. \tag{1.36}$$

jest $G_0(x) := c(y)\mathrm{e}^{-\frac{x^2}{2}}$. Uzmieniając stałą $c(y)$ dostajemy równanie

$$\partial_x c(x,y) = \mathrm{e}^{\frac{x^2}{2}}\delta(x-y) = \mathrm{e}^{\frac{y^2}{2}}\delta(x-y).$$

Stąd

$$G(x,y) = \mathrm{e}^{-\frac{x^2}{2}+\frac{y^2}{2}}\theta(x-y) + g(y)\mathrm{e}^{-\frac{x^2}{2}}\mathrm{e}^{-x}.$$

W szczególności, mamy funkcję Greena retardowaną G_+ i adwansowaną G_-:

$$G_+(x) := \mathrm{e}^{-\frac{x^2}{2}+\frac{y^2}{2}}\theta(x-y),$$
$$G_-(x) := -\mathrm{e}^{-\frac{x^2}{2}+\frac{y^2}{2}}\theta(y-x).$$

Zadanie 35 Rozważmy $\mathbb{R}^3$. Pokazać, że $G(x) := \frac{\mathrm{e}^{-m|x|}}{4\pi|x|}$ jest rozwiązaniem równania

$$(-\Delta + m^2)G(x) = \delta(x).$$

Solution. Metoda 1. Niech

$$G(x) = \frac{1}{(2\pi)^3}\int \hat{G}(k)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k.$$

Wtedy

$$\begin{aligned}\hat{G}(k) &= \frac{1}{(k^2+m^2)},\\ G(x) &= \frac{1}{(2\pi)^3}\int \frac{\mathrm{e}^{\mathrm{i}kx}}{(k^2+m^2)}\mathrm{d}k\\ &= \frac{1}{(2\pi)^2}\int_0^\infty |k|^2\mathrm{d}|k|\int_0^\pi \sin\theta\mathrm{d}\theta\frac{\mathrm{e}^{\mathrm{i}|k||x|\cos\theta}}{(k^2+m^2)}\\ &= \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}|k|\frac{\mathrm{e}^{\mathrm{i}|k||x|}-\mathrm{e}^{-\mathrm{i}|k||x|}}{|k||x|\mathrm{i}(k^2+m^2)}\\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^\infty \mathrm{d}s\frac{s\mathrm{e}^{\mathrm{i}s|x|}}{|x|\mathrm{i}(s^2+m^2)} \;=\; \frac{\mathrm{e}^{-m|x|}}{4\pi|x|}.\end{aligned}$$

Metoda 2. Zastosujemy wzór Greena:

$$\int_\Omega(\Delta f g - f\Delta g) = \int_{\partial\Omega}(\nabla f g - f\nabla g)\mathrm{d}s.$$

Kładziemy $f=\phi$, $g=\frac{\mathrm{e}^{-m|x|}}{4\pi|x|}$, $\Omega=\mathbb{R}^3\backslash K(r)$. Mamy na $\partial K(r)=r^2S^2$ z miarą $r^2\mathrm{d}\omega$, gdzie $\omega\in S^2$.

Sprawdzamy, że poza zerem

$$(-\Delta+m^2)\frac{\mathrm{e}^{-m|x|}}{4\pi|x|}=0.$$

$$\int_{\mathbb{R}^3\backslash K(r)}(-\Delta+m^2)\phi(x)\frac{\mathrm{e}^{-m|x|}}{4\pi|x|}\mathrm{d}x = \int_{S^2}\Big(\partial_r\phi(r,\omega)\frac{\mathrm{e}^{-mr}}{4\pi r}-\phi(r,\omega)\partial_r\frac{\mathrm{e}^{-mr}}{4\pi r}\Big)r^2\mathrm{d}\omega$$

Wreszcie, korzystamy z

$$\partial_r\frac{\mathrm{e}^{-mr}}{r} = -\frac{\mathrm{e}^{-mr}}{r^2}-\frac{m\mathrm{e}^{-mr}}{r},\quad \int_{S^2}\mathrm{d}\omega = 4\pi.$$

Zadanie 36 Niech P będzie wielomianem takim, że $P(\xi)\neq 0$, $\xi\in\mathbb{R}$. Niech $g\in\mathcal{S}(\mathbb{R})$. Pokazać, że istnieje $f\in\mathcal{S}(\mathbb{R})$ taka, że

$$P(\mathrm{i}\partial_x)f = g.$$

Pokazać, że istnieje G zależne tylko od P takie, że to rozwiązanie może być zapisane jako

$$f = G * g.$$

Znaleźć G dla $P(\xi)=\xi^2+m^2$

Problem 1.1 *Rozważmy $l^2(\mathbb{Z})$ z bazą kanoniczną $(\delta_n \ : \ n \in \mathbb{Z})$. Niech $\theta \in \mathbb{R}$. Zdefiniujmy wektory*

$$b_n := \cos(\theta n)\delta_n + \sin(\theta n)\delta_{-n}.$$

Pokazać, że $(b_n \ : \ n \in \mathbb{Z})$ jest bazą ortnormalną.

Problem 1.2 *Policzyć tranformatę Fouriera funkcji*

$$f(x) = \mathrm{e}^{-\frac{3}{4}x^2} \cos x^2.$$

Problem 1.3 *Niech $1 < p < \infty$ i $f \in L^p(\mathbb{R}^3)$. Niech $|x|$ oznacza normę euklidesową wektora $x \in \mathbb{R}^3$. Dla jakiego m funkcja $(1+|x|^2)^{-m} f$ należy do $L^1(\mathbb{R}^d)$?*

Problem 1.4 *Niech d będzie liczbą naturalną. Dla jakiego m następująca funkcja należy do $L^2(\mathbb{R}^d)$:*

(i) $|x|^{-m}$,

(ii) $(1+|x|)^{-m}$,

(iii) $\prod_{i=1}^d (1+|x_i|)^{-m}$.

Problem 1.5 *Dla $t > 0$ kładziemy $g_t(x) := (2\pi t)^{-1/2}\mathrm{e}^{-x^2/2t}$. Znaleźć $g_t * g_s$.*

Wskazówka. $\hat{g}_t(\xi) = \mathrm{e}^{-t\xi^2/2}$.

Problem 1.6 *Dla $t \neq 0$ kładziemy $g_t(x) = (\mathrm{i}x + t)^{-1}$. Znaleźć $g_t * g_s$.*

Wskazówka. $\hat{g}_t(\xi) = 2\pi(\mathrm{sgn}t)\theta(\xi \mathrm{sgn}t)\mathrm{e}^{-t\xi}$.

Problem 1.7 *Pokazać, że* $\mathrm{Span}\{(x+\alpha)^{-1} \ : \ \mathrm{Im}\alpha > 0\}$ *nie jest podprzestrzenią gęstą w $L^2(\mathbb{R})$.*

Problem 1.8 *Niech $g \in L^1(\mathbb{R})$. Pokazać, że operator $Tf := f * g$ jest dobrze zdefiniowany dla $f \in L^2(\mathbb{R})$, jest ograniczony i $\|T\| \leq \|g\|_1$. Czy zawsze $\|T\| = \|g\|_1$?*

Wskazówka. Zastosować transformację Fouriera.

Problem 1.9 *Niech $1 \leq p \leq r \leq q \leq \infty$. Pokazać, że*

$$L^p(\mathbb{R}) \cap L^q(\mathbb{R}) \subset L^r(\mathbb{R}) \subset L^p(\mathbb{R}) + L^q(\mathbb{R}).$$

Problem 1.10 *Dla $m \in \mathbb{R}$, $1 \leq p \leq \infty$ definiujemy*

$$L^p_m(\mathbb{R}) := \{f \ : \ \|(1+|x|)^m f\|_p < \infty\}.$$

Pokazać, że jeśli $r \leq q$, $m > \frac{1}{r} - \frac{1}{q} + k$, to $L^r_k(\mathbb{R}) \supset L^q_m(\mathbb{R})$.

Wskazówka. Wykorzystać uogólnioną nierówność Höldera

$$\|fg\|_r \leq \|f\|_p \|g\|_q, \quad 1 \leq p \leq r \leq q \leq \infty, \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

2 Exercises to Theory of Operators

2.1 Inverses

Problem 2.1 *Let A, B be invertible. Then*

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}. \tag{2.37}$$

Problem 2.2 *Let H be invertible and V has a small enough norm. Then $H+V$ and $(\mathbb{1}+VH^{-1})$ are also invertible and*

$$(H + V)^{-1} = H^{-1}(\mathbb{1} + VH^{-1})^{-1}. \tag{2.38}$$

Besides,

$$\|H^{-1} - (H + V)^{-1}\| \leq c_0\|V\|. \tag{2.39}$$

Solution. If $\|VH^{-1}\| =: c < 1$, then the series

$$\sum_{n=0}^{\infty}(-VH^{-1})^n \tag{2.40}$$

can be dominated bby a convergent geometric series, and hence is convergent. We check that it is the inverse of $\mathbb{1} + VH^{-1}$. But

$$(H + V)^{-1} = \big((\mathbb{1} + VH^{-1})H\big)^{-1} = H^{-1}(\mathbb{1} + VH^{-1})^{-1}. \tag{2.41}$$

Now

$$(H + V)^{-1} - H^{-1} = H^{-1}\sum_{n=1}^{\infty}(-VH^{-1})^n. \tag{2.42}$$

Hence,

$$\|(H + V)^{-1} - H^{-1}\| \leq \|H^{-1}\|^2\|V\|\frac{c}{1 - c}. \tag{2.43}$$

Problem 2.3 *Let $H + z_0\mathbb{1}$ be invertible. Then $H + z\mathbb{1}$ is invertible in a neighborhood of z_0. Besides $z \mapsto (H + z\mathbb{1})^{-1}$ is a differentiable function and*

$$\frac{\mathrm{d}}{\mathrm{d}z}(H - z\mathbb{1})^{-1} = (H - z\mathbb{1})^{-2}. \tag{2.44}$$

Solution. By Problem 2.2, with $V = z - z_0$ the function $z \mapsto (H - z)^{-1}$ is well defined and continuous neoar z_0. By Problem 2.1, we have the *resolvent formula*:

$$(H - z_0 - h)^{-1} - (H - z_0)^{-1} = h(H - z_0 + h)^{-1}(H - z_0)^{-1}. \tag{2.45}$$

Now

$$\frac{\mathrm{d}}{\mathrm{d}z}(H - z_0\mathbb{1})^{-1} = \lim_{h\to 0}(H - z_0 - h)^{-1} - (H - z_0)^{-1} = \lim_{h\to 0}(H - z_0 + h)^{-1}(H - z_0)^{-1}. \tag{2.46}$$

2.2 Finite dimensional matrices

Let A be a linear operator on a finite dimensional space $\mathcal{V}$. Let $\lambda \in \mathbb{C}$. TFAE:

(1) $\det(A - \lambda) = 0$,

(2) There exists $v \neq 0$ s.t. $Av = \lambda v$

(3) $(\lambda - A)$ is not invertible.

The set of such λ is called the spectrum of A and denoted sp$A$.

Problem 2.4 *Let* $\{\lambda_1, \ldots, \lambda_n\} = \mathrm{sp}A$ *Show that if* $v_i \in \mathrm{Ker}(\lambda_i - A)$*, and* $v_1 + \cdots + v_n = 0$*, then* $v_1 = \cdots = v_n = 0$.

Solution. Suppose that this is not true. We can assume that p is the smallest possible number of nonzero $v_i \in \mathrm{Ker}(\lambda_i - A)$ such that $v_1 + \cdots + v_p = 0$. Then

$$0 = \lambda_p(v_1 + \cdots v_p) - A(v_1 + \cdots v_p) = (\lambda_p - \lambda_1)v_1 + \cdots (\lambda_p - \lambda_{p-1})v_{p-1} = 0,$$

which is a contradiction. □

We say that A is diagonalizable if $\sum_{i=1}^{n} \mathrm{Ker}(\lambda_i - A) = \mathcal{V}$. In other words, $\mathcal{V} = \bigoplus_{i=1}^{n} (\mathrm{Ker}(\lambda_i - A)$. We can then define the projection P_i onto $\mathcal{V}_i$ along $\mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_{i-1} \oplus \mathcal{V}_{i+1} \oplus \cdots \oplus \mathcal{V}_n$. We have then

$$A = \sum_{i=1}^{n} \lambda_i P_i.$$

Problem 2.5 *Let* A *be an arbitrary matrix. Let* $\lambda_i \in \mathrm{sp}(A)$*. Show that*

$$\mathrm{Ker}(\lambda_i - A) \subset \cdots \subset \mathrm{Ker}(\lambda_i - A)^m.$$

Show that this sequence stablizes. Suppose this happens for $m_i, m_i + 1, \ldots$*. Set*

$$\mathcal{V}_i := \mathrm{Ker}(\lambda_i - A)^{m_i}.$$

Show that $\mathcal{V}_i$ *is an invariant subspace of* A*.*

****

We have

$$\mathcal{V} = \bigoplus_{i=1}^{n} \mathcal{V}_i, \quad A = \bigoplus_{i=1}^{n} (\lambda_i + N_i),$$

where N_i is nilpotent on $\mathcal{V}_i$. Set

$$D := \bigoplus_{i=1}^{n} \lambda_i, \quad N := \bigoplus_{i=1}^{n} N_i.$$

We have

$$A = D + N, \tag{2.47}$$

where D is diagonalisable, N is nilpotent and $DN = ND$. (2.47) is called *Jordan-Chevalley decomposition*.

Problem 2.6 *Let* $f(z) = \sum_{k=0}^{n} f_k z^k$ *be a polynomial Show that*

$$f(A) := \sum_{k=0}^{n} f_k A^k = \sum \frac{N^j}{j!} f^{(j)}(D). \tag{2.48}$$

Solution. It is enough to assume that $A = \lambda + N$ where N is nilpotent.

$$f(A) = \sum f_k \sum N^j \lambda^{k-j} \frac{k!}{(k-j)!j!} = \sum_{j=0} \frac{N!}{j!} \sum_{j=0}^{m} k(k-1)\cdots(k-j+1) f_k \lambda^{k-j}. \tag{2.49}$$

Note that in order to compute $f(A)$ it is enough to know

$$f(\lambda_i), \cdots f^{(m_i-1)}(\lambda_i),$$

Spectral Theorem. Let $AA^* = A^*A$ (A is normal). Then A is diagonalizable and the spaces $\mathrm{Ker}(\lambda_i - A)$ are orthogonal. Therefore, P_i are orthogonal projections.

If A is Hermitian, then $\mathrm{sp}A \subset \mathbb{R}$.

If A is unitary, then $\mathrm{sp}A \subset \{|z| = 1\}$.

Problem 2.7 *Let* A *be normal. Then*

$$\|A\| = \sup\{|\lambda| \ : \ \lambda \in \mathrm{sp}A\}.$$

In particular

$$\|(z-A)^{-1}\| = \Big(\min\{|z-\lambda| \ : \ \lambda \in \mathrm{sp}A\}\Big)^{-1}.$$

Solution.

$$\|Au\|^2 = \sum |\lambda_i|^2 \|P_i u\|\|^2$$
$$\leq \sup|\lambda_i|^2 \sum \|P_i u\|^2 = (\sup|\lambda_i|)^2 \|u\|^2.$$

Problem 2.8 *Let* A *be any operator with spectrum* $\{\lambda_1, \ldots, \lambda_n\}$*, and degrees of nilpotency* $m_1, \ldots, m_n$*. Then*

$$\|(z-A)^{-1}\| \leq c \sum_{i=1}^{n} |z-\lambda_i|^{-m_i}.$$

Solution.

$$\|(z-A)^{-1}u\| \leq \sum_{i=1}^{n} \|(z-\lambda_i - N_i)^{-1} P_i u\|.$$

$$(z-\lambda_i - N_i)^{-1} = \sum_{j=0}^{m_j - 1} (z-\lambda_i)^{-1-j} N_j.$$

2.3 Operators

Problem 2.9 *Show that* $\|A\| = \|A^*\|$.

Solution.

$$\|A\| = \sup_{\|v\|\leq 1} \|Av\| = \sup_{\|v\|,\|w\|\leq 1} |(w|Av)| = \sup_{\|v\|,\|w\|\leq 1} |(A^*w|v)| = \sup_{\|w\|\leq 1} \|A^*w\| = \|A^*\|.$$

Problem 2.10 *Let* $A, B \in B(\mathcal{H})$. *Prove that* $\mathrm{sp}AB \cup \{0\} = \mathrm{sp}BA \cup \{0\}$.

Solution. Let $z \in \mathrm{rs}AB\backslash\{0\}$.Then

$$z^{-1}(1 + B(z - AB)^{-1}A)(z - BA) = z^{-1}(z - BA) + z^{-1}B(z - AB)^{-1}(1 - AB)A = 1.$$

Hence $z^{-1}(1 + B(z - AB)^{-1}A) = (z - BA)^{-1}$ and $z \in \mathrm{rs}(BA)$.

Problem 2.11 *Rozważmy* $L^2(S^1)$ *z bazą ortonormalną* $e_n(\phi) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}n\phi}$, $n \in \mathbb{Z}$. *Niech* $(c_n \ : \ n \in \mathbb{Z})$ *będzie ciągiem ograniczonym. Zdefinujmy operator*

$$C = \sum_{n=-\infty}^{\infty} c_n |e_n)(e_n|.$$

(i) *Pokazać, że* C *jest ograniczony i ma normę* $\sup\{|c_n| \ : \ n \in \mathbb{Z}\}$.

(ii) *Pokazać, że jeśli* $\sum_{n=-\infty}^{\infty} |c_n| < \infty$, *to* C *posiada jądro całkowe równe*

$$C(\phi, \psi) = \frac{\hat{c}(-\phi + \psi)}{2\pi},$$

gdzie $\hat{c}(\psi) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\psi n} c_n$..

Problem 2.12 *Na* $L^2(S^1)$ *rozważyć operator* P_ϵ *z jądrem całkowym*

$$P_\epsilon(\phi, \psi) = \frac{\sinh \epsilon}{\cosh \epsilon - \cos(\phi - \psi)}.$$

Pokazać, że $\mathrm{s}-\lim_{\epsilon \searrow 0} P_\epsilon = \mathbb{1}$ *i* $\|P_\epsilon\| = 1$.

Wskazówka. *Pokazać, używając bazy ortonormalnej z poprzedniego zadanie, że*

$$P_\epsilon = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\epsilon|n|} |e_n)(e_n|.$$

Problem 2.13 *Niech* $f : [0,\infty[\to [0,1]$ *będzie funkcją ciągłą i malejącą, taką, że* $f(0) = 1$ *i* $\lim_{t\to\infty} f(t) = 0$. *Rozważmy przestrzeń Hilberta* l^2 *z bazą kanoniczną* $(\delta_j \ : \ j \in \mathbb{N})$. *Zdefiniujmy roddzinę operatorów*

$$C_\epsilon := \sum_{j=1}^{\infty} f(\epsilon j)|\delta_j)(\delta_j|.$$

(i) *Pokazać, że funkcja* $[0,\infty[\ni \epsilon \mapsto C_\epsilon$ *jest normowo ciągła na* $]0,\infty[$ *lecz normowo nieciągła w* $\epsilon = 0$.

(ii) *Pokazać, że funkcja* $[0,\infty[\ni \epsilon \mapsto C_\epsilon$ *jest silnie ciągła.*

Problem 2.14 *Mówimy, że* P *jest rzutem, gdy* $P^2 = P$.

(i) *Pokazać, że jeśli* P *jest rzutem niezerowym, to* $\|P\| \geq 1$.

(ii) *Pokazać, że dla każdego* $c \geq 1$ *istnieje rzut na przestrzeni Hilberta taki, że* $\|P\| = c$.
Wskazówka. *Wystarczy rozważać 2-wymiarowe przestrzenie Hilberta.*

(iii) *Pokazać, że jeśli* P *jest rzutem na przestrzeni Hilberta takim, że* $\|P\| = 1$, *to jest to rzut ortogonalny.*

Problem 2.15 *Niech* $(U_n \ : \ n = 1, 2, \dots)$ *będzie ciągiem operatorów unitarnych.*

(i) *Pokazać, że jeśli* $\lim_{n\to\infty} U_n = U$, *to* U *jest unitarny.*

(ii) *Pokazać, że jeśli* $\mathrm{s}-\lim_{n\to\infty} U_n = U$ *i* $\mathrm{s}-\lim_{n\to\infty} U_n^* = U^*$, *to* U *jest unitarny.*

(iii) *Pokazać, że jeśli* $\mathrm{s}-\lim_{n\to\infty} U_n = U$, *to* U *jest izometrią. Podać przykład ciągu operatorów unitarnych, którego silna granica nie jest unitarna.*

(iv) *Pokazać, że jeśli* $\mathrm{w}-\lim_{n\to\infty} U_n = U$, *to* $\|U\| \leq 1$. *Podać przykład ciągu operatorów unitarnych, którego słaba granica jest zerem.*

Problem 2.16 *Niech* $(P_n \ : \ n = 1, 2, \dots)$ *będzie ciągiem rzutów. Niech* $\mathrm{s}-\lim_{n\to\infty} P_n = P$. *Pokazać, że* P *jest rzutem.*

Wskazówka. Można założyć, że $\sup \|P_n\| < \infty$. (Wynika to z Tw. Banacha-Steinhausa i silnej zbieżności ciągu (P_n)).

Problem 2.17 *Niech* A_n *będzie ciągiem samosprzężonych operatorów ograniczonych na przestrzeni Hilberta takich, że* $\mathrm{w}-\lim_{n\to\infty} A_n = A$. *Pokazać, że* A *jest samosprzężony.*

Problem 2.18 *Niech* $(P_n \ : \ n = 1, 2, \dots)$ *będzie ciągiem rzutów ortogonalnych.*

(i) *Niech* $\mathrm{s}-\lim_{n\to\infty} P_n = P$. *Pokazać, że* P *jest rzutem ortogonalnym.*

(ii) *Podać przykład ciągu* $(P_n \ : \ n = 1, 2, \dot{)}$ *rzutów ortogonalnych takich, że* $\mathrm{w}-\lim_{n\to\infty} P_n = \frac{1}{2}\mathbb{1}$.

Problem 2.19 *Dla $f \in L^2([0,\infty[)$ definiujemy*

$$(Tf)(x) := x^{-1} f(x^{-1}).$$

Czy T jest operatorem

(i) *ograniczonym,*

(ii) *unitarnym,*

(iii) *samosprzężonym.*

Problem 2.20 *Niech $\alpha \in \mathbb{C}$, $\mathrm{Re}\alpha \leq 0$. Niech $(\delta_n : n \in \mathbb{N})$ oznacza bazę kanoniczną w $l^2(\mathbb{N})$. Zdefiniujmy następujący operator na $l^2(\mathbb{N})$:*

$$T := \sum_{n=1}^{\infty} \mathrm{e}^{n\alpha} |\delta_{n+1})(\delta_n|.$$

(i) *Pokazać, że T jest operatorem ograniczonym i znaleźć jego normę.*

(ii) *Dla jakich α operator T jest izometrią?*

(iii) *Policzyć T^*T.*

(iv) *Policzyć T^2.*

(v) *Dla jakich α istnieje $\mathrm{s} - \lim_{n\to\infty} T^n$?*

(vi) *Dla jakich α istnieje $\lim_{n\to\infty} T^n$?*

2.4 Przestrzenie Hilberta

Niech $\mathcal{H}$ będzie przestrzenią Hilberta. Będziemy stosować notację dla iloczynu skalarnego podobną do notacji Diraca:

$$(v|w), \quad v, w \in \mathcal{H}.$$

Jedną z jej zalet jest możliwość "oderwania" $(v|$, $|w)$, traktując je jako operatory

$$\mathbb{C} \ni z \mapsto |w)z := wz \in \mathcal{H}, \tag{2.50}$$

$$\mathcal{H} \ni h \mapsto (v|h := (v|h) \in \mathbb{C}. \tag{2.51}$$

Na przykład

$$|w)(v|h = w(v|h).$$

Jeśli $(v|w) = 1$, jest to rzut na w wzdłuż $\mathrm{Ker}(v|$. Jeśli $\|v\| = 1$, to $|v)(v|$ jest rzutem ortogonalnym na v. Jeśli $e_1, \ldots, e_n$ jest bazą ortonormalną, to

$$A = \sum_{i,j=1}^{n} A_{ij} |e_i)(e_j|.$$

Problem 2.21 *In l^2 we define the spaces*

$$W := \{(x_n) \in l^2 \ : x_{2k} = 0,\ k \in \mathbb{N}\}, \tag{2.52}$$

$$Z := \{(x_n) \in l^2 \ : \ x_{2k-1} + \sqrt{k}x_{2k} = 0,\ k \in \mathbb{N}\}. \tag{2.53}$$

Obviously, W and Z are closed. Show that $W + Z$ is dense in l^2 but not closed.

Solution. Let $x \perp W + Z$. Because $x \perp V$, $x_{2k-1} = 0$. Because $x \perp Z$, $x_{2k-1} - \frac{1}{\sqrt{k}}x_{2k} = 0$. Hence $x = 0$. Therefore, $W + Z$ is dense in l^2.

Consider $x \in l^2$, $x_i = \frac{1}{n}$. Let $x = v + z$, $v \in V$, $z \in Z$. Then

$$z_{2k} = \frac{1}{2k}, \quad z_{2k-1} = -\sqrt{k}z_{2k} = -\frac{1}{2\sqrt{k}}.$$

But $z \notin l^2$.

Problem 2.22 *Let A be a self-adjoint operator on $\mathbb{C}^n$. Show*

$$\|A\| = \max\{|\lambda| \ : \ \lambda \in \mathrm{sp}(A)\} = \sup\{(v|Av) \mid \|v\| = 1\}.$$

Solution.

$$A = \sum_{\lambda \in \mathrm{sp}(A)} \lambda P_\lambda.$$

$$\|Av\|^2 = \sum \lambda^2 \|P_\lambda v\| \leq \max\{|\lambda| \ : \ \lambda \in \mathrm{sp}(A)\} \sum \|P_\lambda v\|^2 \tag{2.54}$$

$$= \max\{|\lambda| \ : \ \lambda \in \mathrm{sp}(A)\}\|v\|^2. \tag{2.55}$$

Problem 2.23 $\|B\|^2 = \|B^*B\|$.

Solution. Clearly, B^*B is self-adjoint.

$$\|B\| = \sup\{\|Bv\| \mid \|v\| = 1\} = \sup\{(v|B^*Bv) \mid \|v\| = 1\} = \|B^*B\|.$$

Problem 2.24 *Find the norm of* $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$.

Solution.

$$B^*B = \begin{bmatrix} \overline{B_{11}}B_{11} + \overline{B_{21}}B_{21} & \overline{B_{11}}B_{12} + \overline{B_{21}}B_{22} \\ \overline{B_{12}}B_{11} + \overline{B_{22}}B_{21} & \overline{B_{12}}B_{12} + \overline{B_{22}}B_{21} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$\det(C - \lambda \mathbb{1}) = (C_{11} - \lambda)(C_{22} - \lambda) - C_{12}C_{21} = 0.$$

$$\|B\|^2 = \frac{C_{11} + C_{22} + \sqrt{(C_{11} - C_{22})^2 + 4C_{12}C_{21}}}{2}.$$

2.5 Spectrum

We say that A is involution if $A^2 = \mathbb{1}$.

Problem 2.25 *Show that the following are equivalent:*

(1) *A is self-adjoint and A is an involution.*

(2) *A is an involution and A is unitary*

(3) *A is unitary and A is self-adjoint.*

Problem 2.26 *Let A be an involution. Find its spectrum and its spectral projections.*

Solution. We guess: set $P_\pm := \frac{1}{2}(\mathbb{1} \mp A)$. Then $AP_\pm = \pm P_\pm$ and $A = P_+ - P_-$. Therefore, $\mathrm{sp}(A) = \{-1, 1\}$ and $\mathbb{1}_{\{\pm 1\}}(A) = P_\pm$.

Problem 2.27 *Let $U^n = 1$. Find spectral projections of U.*

Solution. Similarly, as in the previous problem, we guess:

$$P_k = \frac{1}{n}\sum_{j=0}^{n-1} U^j \mathrm{e}^{\frac{-\mathrm{i}jk2\pi}{n}} = \mathbb{1}_{\{\mathrm{e}^{\frac{\mathrm{i}2\pi k}{n}}\}}(U), \tag{2.56}$$

$$U = \sum_{k=0}^{n=1} \mathrm{e}^{\frac{\mathrm{i}k2\pi}{n}} P_k. \quad \mathrm{sp}(U) = \{\mathrm{e}^{\frac{\mathrm{i}2\pi k}{n}} \ : \ k = 0, \dots, n-1\}. \tag{2.57}$$

Problem 2.28 *Find the spectrum of the Fourier transformation $\mathcal{F}$.*

Solution. $\mathcal{F}^4 = \mathbb{1}$. Hence $\mathrm{sp}\mathcal{F} \subset \{1, \mathrm{i}, -1, -\mathrm{i}\}$. Let $\Omega(x) = \mathrm{e}^{-\frac{x^2}{2}}$, $a^* = x - \partial_x$. Then $\mathcal{F}\Omega = \Omega$ and $\mathcal{F}a^* = \mathrm{i}a^*\mathcal{F}$. Hence $\mathcal{F}^n a^{*n}\Omega = \mathrm{i}^n a^{*n}\Omega$. But $a^{*n}\Omega$ is a complete set of eigenvectors of the harmonic oscillator.

2.6 Operator inequalities

We say that $A \in B(\mathcal{H})$ satisfies $A \geq 0$ if

$$(v|Av) \geq 0, \quad v \in \mathcal{H}.$$

Equivalent condition: A is self-adjoint and $\mathrm{sp}A \subset [0, \infty[$.

Let A be self-adjoint. Set $\inf \mathrm{sp}A = a_-$, $\sup \mathrm{sp}A = a_+$. Then

$$a_- \leq A \leq a_+.$$

For any A, $A^*A \geq 0$.

Problem 2.29 *Show that*

$$AA^* \leq \mathbb{1} \quad \Leftrightarrow \quad A^*A \leq \mathbb{1} \quad \Leftrightarrow \quad \|A\| \leq 1. \tag{2.58}$$

Solution. We have

$$\|Av\|^2 = (v|A^*Av) \leq (v|v) = \|v\|^2 \Leftrightarrow \|A\| \leq 1,$$
$$\|A^*v\|^2 = (v|AA^*v) \leq (v|v) = \|v\|^2 \Leftrightarrow \|A^*\| \leq 1.$$

But $\|A\| = \|A^*\|$.

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Let $A, B \in B(\mathcal{H})$. We write $A \leq B$ if

$$(v|Av) \leq (v|Bv), \quad v \in \mathcal{H}.$$

$$A_1 \leq B_1, \quad A_2 \leq B_2 \text{ implies } A_1 + A_2 \leq B_1 + B_2. \tag{2.59}$$

$$A \leq B \text{ implies } CAC^* \leq CBC^*. \tag{2.60}$$

Problem 2.30 *Let $0 \leq A \leq B$. Show that for $t \geq 0$,*

$$(t + B)^{-1} \leq (t + A)^{-1}. \tag{2.61}$$

Solution. Using (2.58), we obtain the following implications:

$$t + A \leq t + B, \tag{2.62}$$
$$(t + B)^{-\frac{1}{2}}(t + A)(t + B)^{-\frac{1}{2}} \leq \mathbb{1}, \tag{2.63}$$
$$(t + A)^{\frac{1}{2}}(t + B)^{-1}(t + A)^{\frac{1}{2}} \leq \mathbb{1}, \tag{2.64}$$
$$(t + B)^{-1} \leq (t + A)^{-1}. \tag{2.65}$$

Problem 2.31 *Let $A \geq 0$ and $\mathrm{Ker}A = \{0\}$. Let $0 < \alpha, \beta < 1$. Then*

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{t^{-\alpha} \mathrm{d}t}{(A+t)}, \tag{2.66}$$

$$A^{\beta} = \frac{\sin \pi\beta}{\pi} \int_0^\infty \Big(\frac{1}{t} - \frac{1}{(A+t)}\Big) t^{\beta} \mathrm{d}t. \tag{2.67}$$

Solution. We start from identity

$$\frac{\pi}{\sin \pi\alpha} = \int_0^\infty \frac{s^{-\alpha} \mathrm{d}s}{(1+s)}. \tag{2.68}$$

We substitute $s = \frac{t}{A}$ to get (2.66). Next we multiply (2.66) by A, use

$$\frac{A}{(t+A)} = \Big(\frac{1}{t} - \frac{1}{(A+t)}\Big) t$$

and set $\beta = 1 - \alpha$, to obtain (2.67).

Problem 2.32 *Let $0 \leq A \leq B$ and $\mathrm{Ker}A = \{0\}$. Let $0 < \alpha < 1$. Then*

$$B^{-\alpha} \leq A^{-\alpha}, \tag{2.69}$$

$$A^{\alpha} \leq B^{\alpha}. \tag{2.70}$$

Solution. We have $(t+B)^{-1} \leq (t+A)^{-1}$. Therefore (2.69) follows from (2.66).

We have $\frac{1}{t} - \frac{1}{(A+t)} \leq \frac{1}{t} - \frac{1}{(B+t)}$. Therefore, (2.70) follows from (2.67).

Problem 2.33 *Find an example of $A \leq B$ such that $A^2 \leq B^2$ is not true.*

Solution. We use the following criterion for positivity:

$$\begin{bmatrix} a & b \\ \overline{b} & d \end{bmatrix} \geq 0 \quad \Leftrightarrow \quad a \geq 0 \text{ and } ad - |b|^2 \geq 0.$$

Set

$$A := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_\epsilon := \begin{bmatrix} 1+\epsilon^2 & \epsilon \\ \epsilon & 1 \end{bmatrix}.$$

Clearly, $A \leq B$. Now

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_\epsilon^2 = \begin{bmatrix} 1+3\epsilon^2+\epsilon^4 & 2\epsilon+\epsilon^3 \\ 2\epsilon+\epsilon^3 & 1+\epsilon^2 \end{bmatrix}.$$

Then

$$B^2 - A^2 = \begin{bmatrix} 3\epsilon^2+\epsilon^4 & 2\epsilon+\epsilon^3 \\ 2\epsilon+\epsilon^3 & 1+\epsilon^2 \end{bmatrix}, \quad \det(B^2 - A^2) = 3\epsilon^2 - 4\epsilon^2 + O(\epsilon^4).$$

Hence $\det(B^2 - A^2) < 0$ for small ϵ.

2.7 Polar decomposition

Let A be an operator such that $\mathrm{Ker}A = \{0\}$ and $\mathrm{Ker}A^* = \{0\}$. Then there exists a unique positive operator, denoted $|A|$ and a unitary operator U such that

$$A = |A|U. \tag{2.71}$$

Besides,

$$|A| = \sqrt{AA^*}, \quad U = |A|^{-1}A.$$

(2.71) is called the *polar decomposition* of A.

The above definition has a generalization to an arbitrary operator. More precisely, if A is arbitrary, then there exists a unique positive operator $|A|$ and a unique partial isometry U such that $\mathrm{Ker}U = \mathrm{Ker}|A|$ and

$$A = U|A|. \tag{2.72}$$

Then (2.72) is called *polar decomposition* of A and $|A| = \sqrt{A^*A}$.

Let B be the inverse of $|A|$ restricted to $\mathrm{Ran}Q$, extended by 0 on $\mathrm{Ker}A$. Then

$$A^+ = BU^*, \tag{2.73}$$

$$|A|^+ = B, \tag{2.74}$$

$$U^+ = U^*. \tag{2.75}$$

Uwaga 2.34 *Let us denote the orthogonal projection onto the closure of* $\mathrm{Ran}A$ *by* P *and onto* $(\mathrm{Ker}A)^\perp$ *by* Q. *The* Moore-Penrose pseudoinverse *is defined as the unique operator* A^+ *such that*

$$AA^+ = P, \quad A^+A = Q. \tag{2.76}$$

Problem 2.35 *Let* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a matrix with determinant* 1. *Prove that the transformation on* $L^2(\mathbb{R})$ *given by*

$$Uf(x) := \frac{1}{|cx+d|} f\left(\frac{ax+b}{cx+d}\right), \quad x \in \mathbb{R}$$

is unitary.

Problem 2.36 *Let* A *be an invertible operator. Then there exists a unique positive operator* B *and unitary* U *such that*

$$A = UB.$$

Solution. We have $A^* = BU^*$, $A^*A = B^2$. Hence $B = \sqrt{A^*A}$. B^2 is invertible. Hence so is B. Therefore, $U = AB^{-1}$. Then we check that U is unitary.

Problem 2.37 *Let* S^1 *be the unit circle parametrized with angle* $\phi \in [0, 2\pi[$. *Let* $S^1 \ni \phi \mapsto \psi(\phi) \in S^1$ *be the bijection of class* C^1 *such that* $\frac{\mathrm{d}\phi}{\mathrm{d}\psi}$ *is bounded. Define the operator* W *on functions on* S^1 *by*

$$Wf(\phi) = f(\psi(\phi))).$$

(i) *Find* W^*.

(ii) *Find the unitary operator* U *and the positive operator* A *such that* $W = BU$.

(iii) *Show that że* W *is bounded on* $L^2(S^1)$ *and find* $\|W\|$.

Solution.

$$W^*g(\psi) = \Big|\frac{\mathrm{d}\phi}{\mathrm{d}\psi}\Big| g\big(\phi(\psi)\big).$$

Hence,

$$W^*Wf(\psi) = \Big|\frac{\mathrm{d}\phi}{\mathrm{d}\psi}\Big| f(\psi), \tag{2.77}$$

$$\sqrt{W^*W}f(\psi) = \Big|\frac{\mathrm{d}\phi}{\mathrm{d}\psi}\Big|^{\frac{1}{2}} f(\psi), \tag{2.78}$$

$$Uf(\phi) = \Big|\frac{\mathrm{d}\psi}{\mathrm{d}\phi}\Big|^{\frac{1}{2}} f(\psi(\phi)). \tag{2.79}$$

Problem 2.38 *Find the polar decomposition of* A *on* $l^2(\mathbb{Z})$ *given by*

$$Ae_n = a_{n+1}e_{n+1},$$

where $a_n \neq 0$, $n \in \mathbb{Z}$.

Solution.

$$|A|e_n = |a_n|e_n, \quad Ue_n = \frac{a_{n+1}}{|a_{n+1}|}e_{n+1}.$$

Problem 2.39 *Find the polar decomposition of* A_z *on* $L^2(\mathbb{R})$ *given by the integral kernel*

$$A_z(x,y) = \mathrm{e}^{-z\frac{(x-y)^2}{2}}, \quad \mathrm{Re}z > 0.$$

Solution. First we compute

$$A_zA_w = \sqrt{\frac{zw2\pi}{z+w}}A_{z+w}.$$

Therefore,

$$|A_z|(x,y) = \sqrt{2\pi\mathrm{Im}z}\mathrm{e}^{-\mathrm{Re}z\frac{(x-y)^2}{2}}, \quad U_z(x,y) = \frac{1}{\sqrt{2\pi\mathrm{Im}z}}\mathrm{e}^{-\mathrm{Im}z\frac{(x-y)^2}{2}}.$$

Problem 2.40 *Let* $\mathbb{R} \ni x \mapsto y(x) \in \mathbb{R}$ *be an increasing bijection. Find the polar decomposition of* A *on* $L^2(\mathbb{R})$

$$Af(x) := f(y(x)).$$

Solution. First we compute

$$A^*g(y) = g(x(y))\left|\frac{\mathrm{d}x}{\mathrm{d}y}(y)\right|.$$

Therefore,

$$|A|f(y) = \left|\frac{\mathrm{d}x}{\mathrm{d}y}(y)\right|^{\frac{1}{2}} f(y),$$

$$Uf(x) = \left|\frac{\mathrm{d}y}{\mathrm{d}x}(x)\right|^{\frac{1}{2}} f(y(x)).$$

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Problem 2.41 (1) *Let U be a unitary operator and P an orthogonal projection. Then $W := UP$ is a partial isometry.*

(2) *Let W be a partial isometry on a finite dimensional Hilbert space. Then there exists a unitary operator U and an orthogonal projection P such that $W = UP$.*

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Problem 2.42 *Find the polar decomposition of $|v)(w|$, where v, w are arbitrary vectors.*

Solution.

$$|v)(w| = \frac{|v)(w|}{\|v\|\|w\|} \cdot \frac{|w)(w|\|v\|}{\|w\|}.$$

2.8 Rank one perturbations

Let us start with a physical example. Consider $l^2(\mathbb{Z})$ with the canonical basis e_n, $n \in \mathbb{Z}$. Consider the Hamiltonian

$$H_0 e_n = e_{n-1} + e_{n+1}, \quad \text{or} \quad (H_0 f)_n = f_{n+1} + f_{n-1},$$

perturbed by λV, where

$$V e_n = \delta_{0,n} e_0, \quad \text{or} \quad (V f)_n = \delta_{0,n} f_0.$$

We would like to find the spectrum of $H = H_0 + \lambda V$.

Introduce the Fourier transformation $\mathcal{F} : l^2(\mathbb{Z}) \to L^2[-\pi, \pi]$

$$(\mathcal{F} e_n)(k) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}nk}.$$

Then

$$\mathcal{F} H_0 \mathcal{F}^{-1} f(k) = 2 \cos k f(k), \quad \mathcal{F} V \mathcal{F}^{-1} = |v)(v|$$

$v(k) = \frac{1}{\sqrt{2\pi}}$. Thus $\sigma(H_0) = [-2, 2]$.

In the sequel we will consider an abstract version of this problem. We assume that H_0 is an operator of multiplication

$$Hf(x) = xf(x),$$

on $L^2[a, b]$ and $v \in L^2[a, b]$. Let

$$Hf = \beta f.$$

Then

$$xf(x) + v(x)\lambda \int \overline{v}(y) f(y) \mathrm{d}y = \beta f(x).$$

Hence

$$f(x) = \frac{\lambda v(x)}{\beta - x} \int \overline{v(y)} f(y) \mathrm{d}y.$$

$$1 = \lambda \int \frac{|v(x)|^2}{\beta - x} \mathrm{d}x.$$

Assume that v is continuous and nonzero on $]a, b[$. Then $\int \frac{|v(x)|^2}{\beta - x} \mathrm{d}x = \infty$ for $\beta \in]a, b[$. We have

$$\frac{\mathrm{d}}{\mathrm{d}\beta} \int \frac{|v(x)|^2}{\beta - x} \mathrm{d}x = -\int \frac{|v(x)|^2}{(\beta - x)^2} \mathrm{d}x < 0, \tag{2.80}$$

$$\lim_{\beta \to \pm\infty} \int \frac{|v(x)|^2}{\beta - x} \mathrm{d}x = 0. \tag{2.81}$$

Set

$$A := \int \frac{|v(x)|^2}{a - x} \mathrm{d}x, \qquad B := \int \frac{|v(x)|^2}{b - x} \mathrm{d}x.$$

Hence on $]-\infty, a[$ we have exactly one eigenvalue for $\lambda \in]-\infty, A^{-1}[$ and on $]b, \infty[$ for $\lambda \in]B^{-1}, \infty[$. We have

$$\lim_{\lambda \to \pm\infty} \frac{\beta(\lambda)}{\lambda} = 1.$$

The eigenvector is

$$\Psi_\lambda(x) = \Big(\int \frac{|v(x)|^2}{(\beta - x)^2} \mathrm{d}x \Big)^{-1} \frac{v(x)}{(\beta - x)}.$$

Let us compute the resolvent:

$$(z - H)^{-1} = (z - H_0)^{-1} + \Big(\lambda^{-1} - (v|(z - H_0)^{-1} v)\Big)^{-1} (z - H_0)^{-1}|v)(v|(z - H_0)^{-1}$$

Hence, by computing the residue of the resolvent at β, we get

$$\mathbb{1}_{\{\beta\}}(H) = (\beta - H_0)^{-1}|v)(v|(\beta - H_0)^{-1} \frac{1}{(v|(\beta - H_0)^{-2} v)}.$$

2.9 Resonances

For $a \in \mathbb{R}$, using z as a real variable, let us first define the distribution on $\mathbb{R}$

$$\frac{1}{(z - a + \mathrm{i}0)} := \lim_{\epsilon \searrow 0} \frac{1}{(z - a + \mathrm{i}\epsilon)}. \tag{2.82}$$

Note that it is a tempered distribution and we can compute its Fourier transform:

$$\frac{1}{2\pi \mathrm{i}} \int \frac{\mathrm{e}^{-\mathrm{i}tz}}{z - a + \mathrm{i}0} \mathrm{d}z = -\mathrm{e}^{-\mathrm{i}ta} \theta(t). \tag{2.83}$$

Indeed,

$$\int \theta(t) \mathrm{e}^{-\mathrm{i}ta} \mathrm{e}^{\mathrm{i}tz} \mathrm{d}t = \lim_{\epsilon \searrow 0} \int_0^\infty \mathrm{e}^{\mathrm{i}t(z - a + \mathrm{i}\epsilon)} \mathrm{d}t \tag{2.84}$$

$$= \lim_{\epsilon \searrow 0} \frac{\mathrm{i}}{z - a + \mathrm{i}\epsilon} = \frac{\mathrm{i}}{z - a + \mathrm{i}0}. \tag{2.85}$$

Let H be a self-adjoint operator. Clearly,

$$\mathbb{C} \backslash \mathrm{sp}(H) \ni z \mapsto (z - H)^{-1}$$

is an analytic function that has poles at points of the discrete spectrum and the residues are the corresponding spectral projections. We cannot extend this function to a larger domain. However, sometimes we can extend

$$z \mapsto (\Phi|(z - H)^{-1} \Psi) \tag{2.86}$$

for some vectors Ψ. The additional domain arising from this extension is sometimes called the "non-physical sheet of the complex plane".

Suppose that and $\mathcal{D}$ a distinguished subspace of $\mathcal{H}$. Suppose that for all $\Psi \in \mathcal{D}$ (2.86) can be extended to some common region Ξ We say that $E \in \Xi$ is a *resonance* if (2.86) has a singularity at E.

Note that for $\epsilon > 0$

$$\int_0^\infty \mathrm{e}^{\mathrm{i}t(z - H + \mathrm{i}\epsilon)} \mathrm{d}t = \frac{\mathrm{i}}{(z - H + \mathrm{i}\epsilon)}. \tag{2.87}$$

The limit of (2.87) for $\epsilon \searrow 0$ does not exist in terms of operators, but for appropriate Φ, Ψ there may exist the limit of matrix elements:

$$\Big(\Phi| \int_0^\infty \mathrm{e}^{\mathrm{i}t(z-H)} \mathrm{d}t \Psi\Big) = \lim_{\epsilon \searrow 0} \Big(\Phi| \int_0^\infty \mathrm{e}^{\mathrm{i}t(z-H+\mathrm{i}\epsilon)} \mathrm{d}t \Psi\Big) \tag{2.88}$$

$$= \lim_{\epsilon \searrow 0} \Big(\Phi| \frac{\mathrm{i}}{(z-H+\mathrm{i}\epsilon)} \Psi\Big) =: \Big(\Phi| \frac{\mathrm{i}}{(z-H+\mathrm{i}0)} \Psi\Big). \tag{2.89}$$

Therefore, (omitting Φ, Ψ) for $t > 0$, applying the inverse Fourier transformation

$$\mathrm{e}^{-\mathrm{i}tH} = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} (z - H + \mathrm{i}0)^{-1} \mathrm{e}^{-\mathrm{i}tz} \mathrm{d}z.$$

By deforming the contour, pushing it down and picking up the residue at E, we obtain for $\Phi, \Psi \in \mathcal{D}$

$$(\Phi|\mathrm{e}^{-\mathrm{i}tH}\Psi) = \frac{1}{2\pi\mathrm{i}} \int_\gamma (\Phi|(z-H)^{-1}\Psi)\mathrm{e}^{-\mathrm{i}tz} + (\Phi|R\Psi)\mathrm{e}^{-\mathrm{i}tE}.$$

2.10 Feshbach-Schur formula

Suppose that the space is $\mathcal{V} = \mathcal{V}_\mathrm{S} \oplus \mathcal{V}_\mathrm{R}$. (S stands for a „small system" and R for a „reservoir"). An operator on $\mathcal{V}$ can be written as

$$H = \begin{bmatrix} H_{\mathrm{SS}} & H_{\mathrm{SR}} \\ H_{\mathrm{RS}} & H_{\mathrm{RR}} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We also introduce the imbeddings J_R and J_S of $\mathcal{V}_\mathrm{R}$, resp. $\mathcal{V}_\mathrm{S}$ into $\mathcal{V}$.
Problem. Write H as

$$H = \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}.$$

Solution

$$H = \begin{bmatrix} 1 & bd^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d^{-1}c & 1 \end{bmatrix}. \tag{2.90}$$

Problem. Compute $J_\mathrm{S}^* H^{-1} J_\mathrm{S}$.

Solution Note that $\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -y \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}^{-1} = \begin{bmatrix} \alpha^{-1} & 0 \\ 0 & \beta^{-1} \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -x & 1 \end{bmatrix}$. Therefore, application of (2.90) yields the inverse of H:

$$H^{-1} = \begin{bmatrix} 1 & 0 \\ -d^{-1}c & 1 \end{bmatrix} \begin{bmatrix} (a - bd^{-1}c)^{-1} & 0 \\ 0 & d^{-1} \end{bmatrix} \begin{bmatrix} 1 & -bd^{-1} \\ 0 & 1 \end{bmatrix}$$

Now $J_\mathrm{S}^* = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $J_\mathrm{S} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Hence

$$J_\mathrm{S}^* H^{-1} J_\mathrm{S} = (a - bd^{-1}c)^{-1}.$$

Applied to $z - H$ instead of H it is sometimes called the *Feshbach-Schur formula*

$$J_\mathrm{S}^*(z-H)^{-1}J_\mathrm{S} = \big(z - H_{\mathrm{SS}} - H_{\mathrm{SR}}(z-H_{\mathrm{RR}})^{-1}H_{\mathrm{RS}}\big)^{-1}. \tag{2.91}$$

2.11 The Friedrichs Hamiltonian

On the Hilbert space $L^2]a,b[\oplus\mathbb{C}$ consider

$$G=\begin{bmatrix}H_0 & |v)\\ (v| & \varepsilon\end{bmatrix}$$

$$H_0f(k)=kf(k).$$

Let us look for an eigenvector of the form (f,g).

$$kf(k)+v(k)g=zf(k) \tag{2.92}$$

$$\int v(k)f(k)+\epsilon g=zg. \tag{2.93}$$

If $g\neq 0$, this yields

$$z=\varepsilon+\int_a^b\frac{|v(k)|^2}{z-k}\mathrm{d}k,\quad f=\frac{v(k)g}{z-k}. \tag{2.94}$$

Now on $]-\infty,a]$ and on $[b,+\infty[$ the function $F(z):=\varepsilon+\int_a^b\frac{|v(k)|^2}{z-k}\mathrm{d}k$ is decreasing. Therefore,

$$\text{if } \varepsilon+\int_a^b\frac{|v(k)|^2}{a-k}\mathrm{d}k<a \text{ there exists a unique eigenvalue in }]-\infty,a[, \tag{2.95}$$

$$\text{if } \varepsilon+\int_a^b\frac{|v(k)|^2}{b-k}\mathrm{d}k<b \text{ there exists a unique eigenvalue in }]b,+\infty[. \tag{2.96}$$

Let us call this eigen value E. Then we have an eigenvector with the eigenprojection

$$\left(1+\int_a^b\frac{|v(k)|^2}{(E-k)^2}\mathrm{d}k\right)^{-1}\begin{bmatrix}\frac{v(k)}{E-k}\\ 1\end{bmatrix}\begin{bmatrix}\overline{\frac{v(k)}{E-k}} & 1\end{bmatrix}. \tag{2.97}$$

For small λ this eigenvalue can be found from

$$E_\lambda\simeq\varepsilon+\lambda^2\int_a^b\frac{|v(k)|^2}{(E-k)}\mathrm{d}k \tag{2.98}$$

(2.97) can be also obtained from the resolvent:

$$\begin{aligned}&(z-G)^{-1}\\ &=\begin{bmatrix}\mathbb{1} & \frac{v(k)}{(z-k)}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{z-k} & 0\\ 0 & \left(z-\varepsilon-\int\frac{|v(k)|^2\mathrm{d}k}{(z-k)}\right)^{-1}\end{bmatrix}\begin{bmatrix}\mathbb{1} & 0\\ \frac{v(k)}{(z-k)} & 1\end{bmatrix}\\ &=\begin{bmatrix}\frac{1}{z-k} & 0\\ 0 & 0\end{bmatrix}\\ &\quad+\begin{bmatrix}\frac{v(k)}{(z-k)}\\ 1\end{bmatrix}\left(z-\varepsilon-\int\frac{|v(k)|^2\mathrm{d}k}{(z-k)}\right)^{-1}\begin{bmatrix}\frac{v(k)}{(z-k)} & 1\end{bmatrix}\end{aligned}$$

Suppose now that $v(k) \neq 0$ for $k \in]a, b[$ and v is continuous. Then for λ small (2.94) has no solutions and hence there are no eigenvalues. However, there may be a resonance. If we replace v with λv, where v is small, then

$$E_\lambda \simeq \varepsilon + \lambda^2 \mathcal{P} \int_a^b \frac{|v(k)|^2}{(E-k)} \mathrm{d}k - \mathrm{i}\pi\lambda^2 |v(E)|^2. \tag{2.99}$$

This is the Fermi Golden Rule I.

This (at least on the heuristic level) implies

$$(\Phi_0 | \mathrm{e}^{-\mathrm{i}tH} \Phi_0) \approx \mathrm{e}^{-\mathrm{i}Et},$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t} \Big| \big(\Phi_0 | \mathrm{e}^{-\mathrm{i}tH} \Phi_0\big) \Big|^2 = 2\pi |v(E)| \Big| \big(\Phi_0 | \mathrm{e}^{-\mathrm{i}tH} \Phi_0\big) \Big|^2,$$

which is called the Fermi Golden Rule II.

Here is an alternative, differential derivation of this rule. Set

$$\begin{bmatrix} \Psi_t \\ \Phi_t \end{bmatrix} := \mathrm{e}^{-\mathrm{i}tG} \begin{bmatrix} 0 \\ \Phi_0 \end{bmatrix}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \Psi_t(k) = -\mathrm{i}k\Psi_t(k) - \mathrm{i}\lambda v(k) \Phi_t, \tag{2.100}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \Phi_t = -\mathrm{i}\lambda \int_a^b \overline{v(k)} \Psi_t(k) \mathrm{d}k - \mathrm{i}\varepsilon \Phi_t. \tag{2.101}$$

Set

$$\tilde{\Psi}_t = \mathrm{e}^{\mathrm{i}tk} \Psi_t, \quad \tilde{\Phi}_t := \mathrm{e}^{\mathrm{i}t\varepsilon} \Phi_t.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{\Psi}_t(k) = -\mathrm{i}\lambda v(k) \mathrm{e}^{\mathrm{i}t(k-\varepsilon)} \tilde{\Phi}_t, \tag{2.102}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{\Phi}_t = -\mathrm{i}\lambda \int_a^b \overline{v(k)} \mathrm{e}^{\mathrm{i}(\varepsilon-k)t} \tilde{\Psi}_t(k) \mathrm{d}k. \tag{2.103}$$

Using the first approximation $\tilde{\Psi}_t(k) = \tilde{\Psi}_0(k) = 0$, $\Phi_t = \Phi_0 = 1$ we obtain after one iteration

$$\tilde{\Psi}_t(k) = -\lambda v(k) \frac{\mathrm{e}^{\mathrm{i}t(k-\varepsilon)} - 1}{k - \varepsilon} \tilde{\Phi}_0.$$

Thus at $t = 0$

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{\Phi}_t = \mathrm{i}\lambda^2 \int_a^b |v(k)|^2 \frac{(1 - \mathrm{e}^{\mathrm{i}t(\varepsilon-k)})}{(k-\varepsilon)} \mathrm{d}k \tilde{\Phi}_t \tag{2.104}$$

$$= \mathrm{i}\lambda^2 \int_a^b |v(\varepsilon - t^{-1}y)|^2 \frac{(\mathrm{e}^{\mathrm{i}y} - 1)}{y} \mathrm{d}y \tilde{\Phi}_t \tag{2.105}$$

$$\approx -\pi\lambda^2 |v(\varepsilon)|^2 \tilde{\Phi}_t. \tag{2.106}$$

where we used

$$\int \frac{\mathrm{e}^{\mathrm{i}y}-1}{y}\mathrm{d}y = \mathrm{i}\pi.$$

Can we have exact exponential decay? Assume $a=-\infty$, $b=\infty$ and $v(k)=\lambda$. Note that formally

$$\mathcal{P}\int \frac{1}{E-k}\mathrm{d}k = -\mathrm{i}\pi|\lambda|^2. \tag{2.107}$$

We have

$$\begin{aligned}&(z-G)^{-1}\\ =&\begin{bmatrix}\frac{1}{z-k} & 0\\ 0 & 0\end{bmatrix}\\ &+\begin{bmatrix}\frac{\lambda}{(z-k)}\\ 1\end{bmatrix}\Big(z-\varepsilon+\mathrm{i}\pi|\lambda|^2\Big)^{-1}\begin{bmatrix}\frac{\lambda}{(z-k)} & 1\end{bmatrix}\end{aligned}$$

Thus

$$\Big(\begin{bmatrix}0\\1\end{bmatrix}|\mathrm{e}^{-\mathrm{i}tG}\begin{bmatrix}0\\1\end{bmatrix}\Big) = \mathrm{e}^{-\mathrm{i}t\epsilon-\pi|\lambda|^2 t}. \tag{2.108}$$

2.12 Perturbation theory

Assume now that H_0 and V are self-adjoint operators on $\mathcal{V}$ and $H_\lambda := H_0+\lambda V$. Set $R_\lambda(z) = (z-H_\lambda)^{-1}$.

Let $\mathcal{V}_{\mathrm{S}}$ be the spectral subspace of H_0 onto the eigenvalue E_0 and $\mathcal{V}_{\mathrm{R}}$ its orthogonal complement. We can write the Feshbach-Schur formula as

$$J_{\mathrm{S}}^* R_\lambda(E) J_{\mathrm{S}} = \Big(E-E_0-\lambda V_{\mathrm{SS}}-\lambda^2 V_{\mathrm{SR}}(z-H_{0\mathrm{RR}}-\lambda V_{\mathrm{RR}})^{-1}V_{\mathrm{RS}}\Big)^{-1}. \tag{2.109}$$

Therefore,

$$\{E\in\mathbb{R}\ :\quad E-E_0-\lambda V_{\mathrm{SS}}-\lambda^2 V_{\mathrm{SR}}(z-H_{0\mathrm{RR}}-\lambda V_{\mathrm{RR}})^{-1}V_{\mathrm{RS}}\quad \text{is non-invertible}\} \tag{2.110}$$

is contained in the spectrum of H.

Problem. Assume that $\dim\mathcal{V}_{\mathrm{S}}$ is finite and E_0 is a discrete eigenvalue of H_0. Find an equation for eigenvalues of H_λ, which for small λ is close to E_0

Solution We can expect that these eigenvalues coincide with (2.110). A finite matrix is non-invertible iff its determinant is zero. Therefore, the condition for these eigenvalues is

$$\det\Big(E-E_0-\lambda V_{\mathrm{SS}}-\lambda^2 V_{\mathrm{SR}}(E-H_{0\mathrm{RR}}-\lambda V_{\mathrm{RR}})^{-1}V_{\mathrm{RS}}\Big)=0. \tag{2.111}$$

Let us remark that we obtain a polynomial in E of degree $\dim\mathcal{V}_{\mathrm{S}}$. In general it has $\dim\mathcal{V}_{\mathrm{S}}$ solutions $\lambda\to E_j(\lambda)$

%%%%%%%%%%%%%%%%%%%%%

Let $\dim \mathcal{V}_{\rm S} = 1$, so that $J_{\rm S} = |\Phi_0)$. Then we expect that close to E_0 there is only one eigenvalue of H_λ. We introduce

$$F_\lambda(E) := E_0 + (\Phi_0|V\Phi_0) + \lambda^2(\Phi_0|V(E - H_{0\rm RR} - \lambda V_{\rm RR})^{-1}V\Phi_0).$$

The eigenvalue E_λ is the solution of

$$E_\lambda = F_\lambda(E_\lambda). \tag{2.112}$$

We can try to solve it by iterations:

$$E_\lambda^{(j)} = F_\lambda^j(E_0). \tag{2.113}$$

The first iteration is

$$E_\lambda^{(1)} = F_\lambda(E_0) = E_0 + (\Phi_0|V\Phi_0) + \lambda^2(\Phi_0|V(E_0 - H_{0\rm RR} - \lambda V_{\rm RR})^{-1}V\Phi_0) \tag{2.114}$$

$$\simeq E_0 + (\Phi_0|V\Phi_0) + \lambda^2(\Phi_0|V(E_0 - H_0)^{-1}_{\rm RR}V\Phi_0) + O(\lambda^3). \tag{2.115}$$

This method of finding eigenvalues is called the *Brillouin-Wigner perturbation theory.*

There is an alternative method, called the *Rayleigh-Schrödinger perturbation theory.* Recall that we have $H = H_0 + \lambda V$, $H_0\Psi_0 = E_0\Psi_0$ and E_0 is a nondegenerate eigenvalue.

We make an ansatz

$$\Psi_\lambda = \sum_{n=0}^{\infty} \lambda^n \Psi_n, \qquad E_\lambda = \sum_{n=0}^{\infty} \lambda^n E_n. \tag{2.116}$$

We assume in addition that

$$(\Psi_0|\Psi_n) = 0, \qquad n = 1, 2, \dots. \tag{2.117}$$

We insert (2.116) into

$$(H_0 + \lambda V)\Psi_\lambda = E_\lambda \Psi_\lambda. \tag{2.118}$$

We obtain a formal series in the powers of λ. At λ^n we have

$$H_0\Psi_n + V\Psi_{n-1} = \sum_{j=0}^{n} E_j \Psi_{n-j}. \tag{2.119}$$

We take the scalar product with Ψ_0:

$$(\Psi_0|H_0\Psi_n) + (\Psi_0|V\Psi_{n-1}) = \sum_{j=0}^{n} E_j(\Psi_0|\Psi_{n-j}). \tag{2.120}$$

With help of (2.117) we simplify (2.120) obtaining

$$(\Psi_0|V\Psi_{n-1}) = E_n. \tag{2.121}$$

(2.118) can be rewritten

$$(E_0 - H_0)\Psi_n = V\Psi_{n-1} - \sum_{j=0}^{n-1} E_j \Psi_{n-j}. \tag{2.122}$$

We multiply (2.122) by $P_{\mathrm{R}} := \mathbb{1} - |\Psi_0)(\Psi_0|$, which does not affect the lhs. Setting

$$R_0' := (E_0 - H_0)^{-1} P_{\mathrm{R}},$$

we obtain

$$\Psi_n = R_0' \Big(V\Psi_{n-1} - \sum_{j=0}^{n-1} E_j \Psi_{n-j} \Big). \tag{2.123}$$

Here are the first iterations:

$$E_1 = (\Psi_0 | V \Psi_0), \tag{2.124}$$
$$\Psi_1 = R_0' V \Psi_0, \tag{2.125}$$
$$E_2 = (\Psi_0 | V R_0' V \Psi_0), \tag{2.126}$$
$$\Psi_2 = R_0' V R_0' V \Psi_0 - (\Psi_0 | V \Psi_0) R_0' V \Psi_0. \tag{2.127}$$

3 Singular perturbations

Let H_0 be a self-adjoint operator of multiplication by k^2 on $L^2(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$. Let h be a function on $\mathbb{R}^d$. Recall that if $h \in L^2(\mathbb{R}^d)$, then the operators $(h|$ and $|h)$ are defined by

$$\begin{aligned} &\mathcal{H} \ni v \mapsto (h|v := (h|v) \in \mathbb{C}, \\ &\mathbb{C} \ni \alpha \mapsto |h)\alpha := \alpha h \in \mathcal{H}. \end{aligned} \tag{3.128}$$

In particular, $|h)(h|$ equals the orthogonal projection onto h times $\|h\|^2$.

$$H_\lambda := H_0 + \lambda |h)(h|, \tag{3.129}$$

is a rank one perturbation of H_0.

We would like to describe how to define (3.129)if h is not necessarily a bounded functional on $\mathcal{H}$. It will turn out that it is natural to consider 3 types of h:

$$\text{I.} \quad \int |h(k)|^2 \mathrm{d}k < \infty, \tag{3.130}$$
$$\text{II.} \quad \int |h(k)|^2 (1+k^2)^{-1} \mathrm{d}k < \infty, \ \int |h(k)|^2 \mathrm{d}k = \infty, \tag{3.131}$$
$$\text{III.} \quad \int |h(k)|^2 (1+k^2)^{-2} \mathrm{d}k < \infty, \ \int |h(k)|^2 (1+k^2)^{-1} \mathrm{d}k = \infty. \tag{3.132}$$

Clearly, in the case I H_λ is self-adjoint on $\mathrm{Dom}H_0$. We will see that in the case II one can easily define H_λ as a self-adjoint operator, but its domain is no longer equal to $\mathrm{Dom}H_0$. In the case III, strictly speaking, the formula (3.129) does not make sense. Nevertheless, it is possible to define a *renormalize* the interaction This procedure resembles the *renormalization of the charge* in quantum field theory. In this case usually the parameter λ looses its meaning, so we will abandon the notation H_λ. Instead, one can label the Hamiltonian by various parameters, which we will put in brackets.

This we have already computed:

Lemat 3.1 *In Case I with $\lambda \neq 0$, the resolvent of H_λ equals*

$$\begin{aligned} R_\lambda(z) &:= (z - H_\lambda)^{-1} \\ &= (z - H_0)^{-1} - g(z)^{-1}(z - H_0)^{-1}|h)(h|(z - H_0)^{-1}, \end{aligned} \tag{3.133}$$

where

$$g(z) := -\lambda^{-1} + (h|(z - H_0)^{-1}h). \tag{3.134}$$

defined for $z \notin \mathrm{sp}H_0$.

Lemat 3.2 *Assume I or II. Then $R_\lambda(z)$ is well defined for $z \in \mathbb{R}\backslash[0, \infty[$, $g_\lambda(z) \neq 0$ and satisfies*

$$R_\lambda(z_1) - R_\lambda(z_2) = (z_2 - z_1)R_\lambda(z_1)R_\lambda(z_2), \tag{3.135}$$

$$\text{or equivalently } \partial_z R_\lambda(z) = -R_\lambda(z)^2. \tag{3.136}$$

Dowód. In Case I (3.135) is obvious. In Case II we see that $(z-H_0)^{-1}h \in L^2$ and $(h|(z-H_0)^{-1}h)$ are well defined. So $R_\lambda(z)$ is well defined and (3.135) extends from Case I to Case II by continuity. □

Assume Case III. Then $R_\lambda(z)$ is ill defined because $(h|(z - H_0)^{-1}h)$ is ill defined. Let us select $c < 0$ and set

$$\partial_z g_{c,\gamma}(z) = -\left(h|(z - H_0)^{-2}h\right), \tag{3.137}$$

$$g_{c,\gamma}(c) = \gamma. \tag{3.138}$$

This can be solved:

$$g_{c,\gamma}(z) = \gamma + (c - z)\Big(h|(z - H)^{-1}(c - H_0)^{-1}h\Big). \tag{3.139}$$

Lemat 3.3 *$g_{c,\gamma}$ defined above satisfies* (3.135) *and* (3.136).

Twierdzenie 3.4 *Assume Case I, II or III except for Case I $\lambda = \infty$. Then for $z \in \mathbb{R}\backslash[0, \infty[$, $g_\lambda(z) \neq 0$ we have* $\mathrm{Ker}R(z) = \{0\}$ *and* $\mathrm{Ran}R(z)$ *is dense. Therefore, $R(z)^{-1}$ is a densely defined operator. Moreover,*

$$H := -R(z)^{-1} + z \tag{3.140}$$

does not depend on z and defines a self-adjoint operator.

4 Cut-off method

Another way to define H for in Case III is the cut-off method. For $\Lambda > 0$ we define

$$h_\Lambda := \mathbb{1}_{[-\Lambda,\Lambda]}(H_0)\, h, \tag{4.141}$$

where $\mathbb{1}_{[-\Lambda,\Lambda]}(H_0)$ is the spectral projection for H_0 onto $[-\Lambda, \Lambda] \subset \mathbb{R}$. Note that h_Λ is square integrable

We fix the *running coupling constant* by

$$g_\Lambda(c) = \gamma. \tag{4.142}$$

Let the *cut-off Hamiltonian* be

$$H_\Lambda := H_0 + \lambda_\Lambda |h_\Lambda)(h_\Lambda|. \tag{4.143}$$

Then the resolvent for H_Λ is given by

$$R_\Lambda(z) = (z - H_0)^{-1} - g_\Lambda(z)^{-1}(z - H_0)^{-1}|h_\Lambda)(h_\Lambda|(z - H_0)^{-1}, \tag{4.144}$$

where

$$g_\Lambda(z) := -\lambda_\Lambda^{-1} + \left(h_\Lambda|(z - H_0)^{-1}h_\Lambda\right). \tag{4.145}$$

The cut-off Hamiltonian converges to the renormalized Hamiltonian:

Twierdzenie 4.1 *Assume* $(1 + k^2)^{-1}h \in L^2$ *(Case I, II or III). Then* $\lim_{k\to\infty} R_\Lambda(z) = R(z)$.

5 Laplacian on $L^2(\mathbb{R}^d)$ with the delta potential

On $L^2(\mathbb{R}^d)$ we consider the unitary operator $U = (2\pi)^{d/2}\mathcal{F}$, where $\mathcal{F}$ is the Fourier transformation. Note that U is unitary.

Let Δ be the usual Laplacian. Clearly,

$$-U\Delta U^* = k^2.$$

Let $|\delta)(\delta|$ be the quadratic form given by

$$(f_1|\delta)(\delta|f_2) = \overline{f_1(0)}f_2(0).$$

Note that again it can be also written as

$$\int \overline{f(x)}\delta(x)g(x)\mathrm{d}x,$$

and thus is interpreted as a "potential". Let $(1|$ denote the functional on $L^2(\mathbb{R}^d)$ given by

$$(1|g) = \int g(k)\mathrm{d}k.$$

Using $\delta(x) = (2\pi)^{-d}\int \mathrm{e}^{\mathrm{i}kx}\mathrm{d}x$ we deduce that

$$U|\delta)(\delta|U^* = (2\pi)^{-d}|1)(1|.$$

Consider

$$U\left(-\Delta + \lambda|\delta)(\delta|\right)U^* = k^2 + \lambda(2\pi)^{-d}|1)(1|$$

as an example of a Hamiltonian with a singular rank one perturbation. We compute:

$$\int \frac{\mathrm{d}^d k}{1+k^2} < \infty \;\Leftrightarrow\; d=1,$$
$$\int \frac{\mathrm{d}^d k}{(1+k^2)^2} < \infty \;\Leftrightarrow\; d=1,2,3,$$
$$\int \frac{\mathrm{d}^d k}{k^2(1+k^2)} < \infty \;\Leftrightarrow\; d=3.$$

Thus

(1) for $d=1$ it is of type IIb, so it can be defined in the form sense using the parameter λ (as we have already seen),

(2) for $d=2$ it is of type IIIb. It can be renormalized.

(3) for d=3 it is of type IIIa. It can be renormalized.

(4) for $d \geq 4$ there is no nontrivial renormalization procedure.

Consider dimension $d=2$. Let us compute the resolvent for $z=-p^2$. We have

$$\begin{aligned} g_{-1,\gamma}(-p^2) &= \gamma + (p^2-1)\frac{(1|(H_0+p^2)^{-1}(H_0+1)^{-1}|1)}{(2\pi)^2} \\ &= \gamma + (p^2-1)\int \frac{\mathrm{d}^2 k}{(2\pi)^2(k^2+p^2)(k^2+1)} \;=\; \gamma + \frac{\ln p^2}{4\pi}. \end{aligned}$$

Using that the Fourier transform of $k \mapsto \frac{1}{k^2+p^2}$ equals $x \mapsto 2\pi K_0(p|x|)$, where K_0 is the 0th MacDonald function, we obtain the following expression for the integral kernel of $(p^2+H)^{-1}$:

$$\frac{1}{2\pi}K_0(p|x-y|) + \frac{K_0(p|x|)K_0(p|y|)}{(2\pi)^2(\gamma_{-1}+\frac{\ln p^2}{4\pi})}. \tag{5.146}$$

In the physics literature one usually introduces the parameter $a = \mathrm{e}^{\gamma/2\pi}$ called the *scattering length*. There is a bound state $K_0(|x|/a)$ with eigenvalue $-a^{-2}$.

Note that

$$\{f \in (1-\Delta)^{-1}L^2(\mathbb{R}^2) :\; f(0)=0\} \tag{5.147}$$

is a closed subspace of $(1-\Delta)^{-1}L^2(\mathbb{R}^2)$. The domain of H is spanned by (5.147) and

$$(-a^{-2}-\Delta)^{-1}|1), \tag{5.148}$$

which is in $L^2(\mathbb{R}^2)\backslash(1-\Delta)^{-1}L^2(\mathbb{R}^2)$. In the position representation (5.148) is $x \mapsto 2\pi K_0(|x|/a)$ Around $r \sim 0$ we have the asymptotics $K_0(r) \simeq -\log(r/2) - \gamma$. Therefore, the domain of H contains functions that behave at zero as $C\big(\log(|x|/2a)+\gamma\big)$.

Consider dimension $d=3$. Let us compute the resolvent for $z=-p^2$. We have

$$\begin{aligned} g_{0,\gamma}(-p^2) &= \gamma + p^2\frac{(1|(H_0+p^2)^{-1}H_0^{-1}|1)}{(2\pi)^3} \\ &= \gamma + p^2\int \frac{\mathrm{d}^3 k}{(2\pi)^3(k^2+p^2)k^2} \;=\; \gamma + \frac{p}{4\pi} \end{aligned}$$

Using that the Fourier transform of $k \mapsto \frac{1}{k^2+p^2}$ equals $x \mapsto 2\pi^2 \frac{\mathrm{e}^{p|x|}}{|x|}$, we obtain the following expression for the integral kernel of $(p^2+H)^{-1}$:

$$\frac{\mathrm{e}^{-p|x-y|}}{4\pi|x-y|} + \frac{\mathrm{e}^{-p|x|}\mathrm{e}^{-p|y|}}{(4\pi)^2(\gamma_0 + \frac{p}{4\pi})|x||y|}. \tag{5.149}$$

In the physics literature one usually introduces the parameter $a = -(4\pi\gamma)^{-1}$ called the *scattering length*.

$$\{f \in (1-\Delta)^{-1}L^2(\mathbb{R}^3) : \ f(0) = 0\} \tag{5.150}$$

is a closed subspace of $(1-\Delta)^{-1}L^2(\mathbb{R}^3)$. The domain of H is spanned by (5.150)

$$(a\mathrm{e}^{\mathrm{i}\pi/4} - \mathrm{i})(\mathrm{i}-\Delta)^{-1}|1) + (a\mathrm{e}^{-\mathrm{i}\pi/4} + \mathrm{i})(-\mathrm{i}-\Delta)^{-1}|1) \tag{5.151}$$

In the position representation $(\pm\mathrm{i}-\Delta)^{-1}|1)$ equals $x \mapsto 2\pi^2 \frac{\exp(\mathrm{e}^{\pm\mathrm{i}\pi/4}|x|)}{|x|}$. Therefore, the Hamiltonian with the scattering length a has the domain whose elements around zero behave as $C(1-a/|x|)$.

For $a > 0$ there is a bound state $\frac{\mathrm{e}^{-|x|/a}}{|x|}$ with eigenvalue $-a^{-2}$. To get the domain, instead of (5.151), we can adjoin this bound state to (5.150).

Note that the Hamiltonian is increasing wrt $\gamma_0 \in]-\infty, \infty]$. It is also increasing wrt a separately on $[-\infty, 0]$ and $]0, \infty]$. At 0 the monotonicity is lost. $a = 0$ corresponds to the usual Laplacian.

The following theorem summarizes a part of the above results.

Twierdzenie 5.1 *Consider* $-\Delta$ *on* $C_{\mathrm{c}}^{\infty}(\mathbb{R}^d \backslash \{0\})$

(1) *It has the defficiency index* $(2,2)$ *for* $d = 1$.

(2) *It has the defficiency index* $(1,1)$ *for* $d = 2, 3$.

(3) *It is essentially self-adjoint for* $d \geq 4$.

(4) *For* $d = 1$ *its Friedrichs extension is* $-\Delta_{\mathrm{D}}$ *and its Krein extension is* $-\Delta$.

(5) *For* $d = 2$ *its Friedrichs and Krein extension is* $-\Delta$.

(6) *For* $d = 3$ *its Friedrichs extension is* $-\Delta$ *an its Krein extension corresponds to* $a = \infty$.