

PHONON DISPERSION RELATION IN WEAKLY COUPLED BOSE GAS BEYOND BOGOLIUBOV APPROXIMATION

J. DEREZIŃSKI AND L. PETTINARI

ABSTRACT. We study the quasiparticle dispersion relation for a homogeneous Bose gas in a large cubic box with periodic boundary conditions at zero temperature. Inspired by the Bogoliubov method, we rearrange the usual Bose gas Hamiltonian in a form adapted to perturbation theory in the weak-coupling regime, both in the canonical and grand-canonical setting. We compute the first non-trivial corrections to the dispersion relation in both settings. We show that for small momenta they are linear in the absolute value of the momentum. Thus in this approximation phonons have a well-defined velocity, which we compute.

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1. INTRODUCTION

Modern developments in the theory of Bose gases largely depend on the original argument proposed by Bogoliubov in 1947 [8]. Within his approach, the Bose gas is approximated by a gas of non-interacting quasiparticles, whose *dispersion relation* is given by the famous formula¹

$$\omega_{\text{bg}}(\mathbf{k}) \equiv \omega_{\text{bg}}(\mathbf{k}, \nu) := \sqrt{\frac{|\mathbf{k}|^4}{4} + \nu \frac{\hat{v}(\mathbf{k})}{\hat{v}(\mathbf{0})} |\mathbf{k}|^2}. \quad (1.1)$$

Here, $\hat{v}(\mathbf{k})$ is the Fourier transform of the interaction potential $v(\mathbf{x})$ and ν is the parameter involved in the so-called “c-number substitution”. (1.1) predicts a linear dispersion relation for *phonons*, quasiparticles with low momentum $|\mathbf{k}|$. For larger momenta, for some shapes of $\hat{v}$, we have a local maximum and next a local minimum—traditionally the corresponding quasiparticles are called *maxons*, resp. *rotons*. It is remarkable that this picture, at least qualitatively, is validated by experiments with ^4He [31, 13] and alkali gases [53]. Neutron scattering (for larger momenta) and sound propagation (for small momenta) in ^4He allow us to measure its quasiparticle dispersion relation, which turns out to be a rather well-defined curve similar to its theoretical prediction. Experiments show that this relation is remarkably sharp. In ^4He we even see maxons and rotons; in alkali gases they are typically absent.

In the strict Bogoliubov approximation the quasiparticle dispersion relation has a precise mathematical meaning: the *Bogoliubov Hamiltonian* can be understood as a system of independent harmonic oscillators with energy described by (1.1). The Bogoliubov Hamiltonian has a well-defined thermodynamic limit, and in this limit $\mathbf{k} \mapsto \omega_{\text{bg}}(\mathbf{k})$ describes a singular shell in the energy-momentum spectrum.

Clearly, the dispersion relation (1.1) is only an approximation to the true one. Mathematically, this is due to two factors:

- (1) To obtain the Bogoliubov Hamiltonian one needs to discard various terms from the full Hamiltonian.
- (2) It is not obvious how to fix the parameter ν , which appears in the Bogoliubov dispersion relation.

In our paper we compute leading corrections to the dispersion relation for weak potentials. More precisely, we replace v with κv , obtaining a perturbative expansion of the energy correction with the small parameter $\sqrt{\kappa}$. We compute the lowest order correction, which has the order κ^1 . We prove that for low momenta this correction is asymptotically linear. Thus in this approximation phonons have a well defined velocity, which we compute.

The first problem that we encounter is the definition of the dispersion relation itself. For the full system in finite volume it is ill defined—it is expected to emerge only in thermodynamic limit. Unfortunately, we do not know how to interpret the full N -body Hamiltonian in thermodynamic limit if we keep the density positive. In

¹We will omit most of the time the dependence on ν in $\omega_{\text{bg}}(\mathbf{k}, \nu)$ and restore it when relevant to the discussion.

the traditional approach of theoretical physics this problem is resolved by considering the Hamiltonian in a finite box of size L with periodic boundary conditions. Then one considers its *Green functions* (time-ordered correlation functions of some observables) and takes their limit as $L \rightarrow \infty$. One expects to see the quasiparticle dispersion relation in singularities of the Green functions analytically continued in energy-momentum. This approach was pioneered by Beliaev [1, 2], who computed the correction to the dispersion relation in the dilute limit.

In our paper we do not introduce Green functions. Instead, we consider the lowest order perturbation theory for one-quasiparticle states in fixed momentum sectors in thermodynamic limit. As explained in [23], this is consistent with the approach based on Green functions.

Already in the lowest nontrivial order of perturbation theory the shift in the dispersion relation has a small nonzero imaginary contribution. Thus quasiparticles are unstable, and the measured dispersion relation should have a nonzero breadth. The imaginary shift of the dispersion relation was first computed by Beliaev in [2], and recently confirmed by one of the authors (JD), Napiórkowski and Ben Li [21].

In this paper we are interested in the real part of the shift of dispersion relation. We will consider it in two settings: *canonical* and *grand-canonical*. We will treat the canonical setting as the main one of our paper, even though computations in the grand-canonical setting seem to be cleaner. However, interestingly, the final result is simpler in the canonical setting.

In the canonical formalism, we fix the density n and we consider a sequence of systems with $n = \lim_{L \rightarrow \infty} \frac{N}{L^d}$, where $N = N(L)$ is the number of particles and L^d is the volume of the box with periodic boundary conditions. The parameter in the initial Bogoliubov approximation is taken to be

$$\nu = \kappa n \hat{v}(\mathbf{0}) = \kappa \lim_{L \rightarrow +\infty} \frac{N \hat{v}(\mathbf{0})}{L^d}. \quad (1.2)$$

In the grand canonical formalism, the number operator N^L is not a constant times the identity, and the main parameter is μ , the *chemical potential*. In the Bogoliubov Hamiltonian the role of the parameter ν is taken by μ_0 defined as

$$\mu_0 = \kappa n_{0,\text{bg}} \hat{v}(\mathbf{0}) = \kappa \left\langle \frac{N_0}{L^d} \right\rangle_{\text{bg}} \hat{v}(\mathbf{0}), \quad (1.3)$$

where $\langle \cdot \rangle_{\text{bg}}$ is the mean value in the Bogoliubov vacuum and N_0 is the zero-mode number operator. In order to compare the two approaches we use the identification

$$\nu = \kappa n_{\text{bg}} \hat{v}(\mathbf{0}) = \kappa \left\langle \frac{N^L}{L^d} \right\rangle_{\text{bg}} \hat{v}(\mathbf{0}). \quad (1.4)$$

The chemical potential μ can be approximately computed from μ_0 . Thus we have three distinct parameters: ν , μ , μ_0 : they have the same leading value, but differ by a term of order κ .

One of the essential ingredients in Bogoliubov's paper is the following idea. When we decompose the total number of particles into the zero-mode and excited modes

$$N^L = N_0 + N^>, \quad N^> = \sum_{\mathbf{p} \neq \mathbf{0}} a_{\mathbf{p}}^* a_{\mathbf{p}},$$

we expect that N_0 is macroscopic, whereas $N^>$ is of lower order. Following the paper of Bogoliubov, this is usually implemented by using the *c-number substitution*, where the zero-mode operators a_0, a_0^* are replaced by the operator $\sqrt{N_0}$, which is then taken to be approximately constant. This procedure is based on the idea that, if the gas is in a condensate state, the zero-mode should be macroscopically occupied, and commutators involving a_0, a_0^* should be negligible in the thermodynamic

limit $N, L^d \rightarrow +\infty$. In our paper we never use a direct c -number substitution. Instead, we treat the Bogoliubov Hamiltonian as the unperturbed part of the full Hamiltonian. This is done in a different way in the canonical and grand-canonical settings.

In our analysis we impose appropriate regularity hypotheses on the interaction potential v and the parameter ν , see Assumptions 2.1 and 3.1. Besides, when we justify discarding some terms from the Hamiltonian in the canonical setting we assume the positivity of the condensate density $n_{0,\text{bg}} > 0$, (which holds automatically for a small coupling constant κ) see Assumption 2.3.

Let us sketch our computations of the correction to the dispersion relation in the canonical setting.

- (1) First we embed the physical space with exactly N particles in a larger space where the zero-mode is allowed to take negative values. This trick goes back to Arnowitt and Girardeau [29]; see also Takano [52]. By separating the zero-mode, one identifies the N -particle space with a truncated Fock space of excitations. This approach was used by Dereziński and Napiórkowski [22], and in a closely related form by Lewin, Nam, Serfaty, and Solovej [39]. It allows us to use the formalism of 2nd quantization and Fock spaces.
- (2) We decompose the finite-volume interaction into its diagonal and off-diagonal parts, V_{diag} and V_{off} ; see Sections 2.3 and 2.5. The diagonal part depends on the zero mode only through N_0 , whereas the off-diagonal part contains a_0 and a_0^* separately. We treat the latter using the extended-space formalism of Section 2.4, obtaining

$$H^L = H_{\text{bg}}^L + \sqrt{\kappa} H_3^L + \kappa H_2^L + \kappa \sum_{i=1}^3 H_{4,i}^L + \kappa \sqrt{\kappa} H_{3,x}^L \quad (1.5)$$

$$+ \kappa R_{4,1}^L + \kappa R_{4,2}^L. \quad (1.6)$$

Here κ is introduced by rescaling $\hat{v} \rightarrow \kappa \hat{v}$ while keeping ν fixed. The operators H_{bg}^L , H_2^L , H_3^L , and $H_{4,i}^L$ are independent of κ ; $H_{3,x}^L$ retains a further κ -dependence. Apart from the remainders $R_{4,i}^L$, these operators act on the excitation Fock space, made over the space of nonzero modes.

- (3) A *Bogoliubov transformation* diagonalizes the quadratic Bogoliubov Hamiltonian H_{bg}^L . Its ground state is Ω_{bg} , and its one-quasiparticle states $b_{\mathbf{k}}^* \Omega_{\text{bg}}$ have excitation energy $\omega_{\text{bg}}(\mathbf{k})$.

We use H_{bg}^L as the unperturbed Hamiltonian in the Rayleigh–Schrödinger perturbation theory [49, Chap. XII]. At the first nontrivial order, κ , the dispersion correction consists of two contributions: The quartic terms gives the *Hellmann-Feynman term*

$$\begin{aligned} \delta_4^{\text{can},L}(\mathbf{k}, \nu, \hat{v}) &:= (b_{\mathbf{k}}^* \Omega_{\text{bg}} | H_4^L b_{\mathbf{k}}^* \Omega_{\text{bg}}) - (\Omega_{\text{bg}} | H_4^L \Omega_{\text{bg}}) \\ &= (\Omega_{\text{bg}} | [b_{\mathbf{k}}, [H_4^L, b_{\mathbf{k}}^*]] \Omega_{\text{bg}}). \end{aligned} \quad (1.7)$$

Subtracting the ground-state correction removes the common extensive contribution. In the thermodynamic limit, we set

$$\delta_4^{\text{can}}(\mathbf{k}, \nu, \hat{v}) := \lim_{L \rightarrow +\infty} \delta_4^{\text{can},L}(\mathbf{k}, \nu, \hat{v}). \quad (1.8)$$

The cubic term yields the *Fermi golden rule contribution*. It has a real and imaginary part. Its imaginary part was computed in [23]; here we study

$$\delta_3^{\text{can}}(\mathbf{k}, \nu, \hat{v}) := \lim_{\epsilon \searrow 0} \lim_{L \rightarrow \infty} \delta_3^{\text{can}, L}(\mathbf{k}, \nu, \hat{v}, \epsilon), \quad (1.9)$$

$$\begin{aligned} \delta_3^{\text{can}, L}(\mathbf{k}, \nu, \hat{v}, \epsilon) := & - (H_3^L b_{\mathbf{k}}^* \Omega_{\text{bg}} | (H_{\text{bg}}^L - E_{\text{bg}}^L - \omega_{\text{bg}}(\mathbf{k}) - i\epsilon)^{-1} H_3^L b_{\mathbf{k}}^* \Omega_{\text{bg}}) \\ & + (H_3^L \Omega_{\text{bg}} | (H_{\text{bg}}^L - E_{\text{bg}}^L - i\epsilon)^{-1} H_3^L \Omega_{\text{bg}}), \end{aligned} \quad (1.10)$$

where E_{bg}^L is the ground-state energy of H_{bg}^L . This expression also admits a commutator representation; see [23, Sec. 2.10] and (2.99). The full correction is the sum:

$$\delta^{\text{can}}(\mathbf{k}, \nu, \hat{v}) := \delta_4^{\text{can}}(\mathbf{k}, \nu, \hat{v}) + \delta_3^{\text{can}}(\mathbf{k}, \nu, \hat{v}). \quad (1.11)$$

- (4) Subsection 2.12, and Section 5, justify discarding the remaining terms. The contributions of $R_{4,i}^L$, H_2^L , $H_{4,1}^L$, and $H_{4,3}^L$ to the order- κ excitation-energy shift vanish in the thermodynamic limit, while $\kappa\sqrt{\kappa}H_{3,x}^L$ contributes only at higher orders.
- (5) We compute the asymptotics as $\frac{|\mathbf{k}|}{\sqrt{\nu}} \rightarrow 0$ of $\delta_4^{\text{can}}(\mathbf{k}, \nu, \hat{v})$ and $\delta_3^{\text{can}}(\mathbf{k}, \nu, \hat{v})$. Adding them up we obtain the final result.

The first main result of our paper is the computation of the phonon velocity correction in the canonical setting:

Theorem 1.1. *In the canonical formalism, the dispersion relation computed up to the order κ has the following low momentum behaviour:*

$$\begin{aligned} & \omega_{\text{bg}}(\mathbf{k}) + \kappa \delta^{\text{can}}(\mathbf{k}, \nu, \hat{v}) \\ & = |\mathbf{k}| c_{\text{can}}(\nu, \hat{v}) \left(1 + o \left(\left(|\mathbf{k}| \nu^{-\frac{1}{2}} \right)^0 \right) \right) \text{ as } \frac{|\mathbf{k}|}{\sqrt{\nu}} \rightarrow 0, \end{aligned} \quad (1.12)$$

where

$$c_{\text{can}}(\nu, \hat{v}) = \sqrt{\nu} - \kappa \frac{\hat{v}(\mathbf{0})\sqrt{\nu}}{16} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{\hat{v}(\mathbf{p})^2}{\hat{v}(\mathbf{0})^2} \frac{|\mathbf{p}|^4}{\omega_{\text{bg}}(\mathbf{p})^3}. \quad (1.13)$$

The computations in the grand-canonical setting are a little different:

- (1) We work in the 2nd quantized formalism from the beginning, without fixing the particle number. The basic object is the *grand-canonical Hamiltonian* $H^L - \mu N^L$, where N^L is the number operator. As already mentioned, the basic parameter is the chemical potential μ .
- (2) We separate the macroscopic zero-mode occupation from its fluctuations using the displacement operator of the zero-mode W_α :

$$W_\alpha^* a_{\mathbf{0}} W_\alpha = a_{\mathbf{0}} - \alpha, \quad \alpha = \sqrt{\frac{\mu_0 L^d}{\kappa \hat{v}(\mathbf{0})}}.$$

The corresponding Bogoliubov state $\Omega_{\text{bg}}(\mu_0)$ satisfies $(a_{\mathbf{0}} - \alpha)\Omega_{\text{bg}}(\mu_0) = 0$, so that its condensate density is $n_{0,\text{bg}} = \frac{\mu_0}{\kappa \hat{v}(\mathbf{0})}$.

- (3) At fixed chemical potential μ , we choose μ_0 by approximately minimizing the expectation of $H^L - \mu N^L$ in $\Omega_{\text{bg}}(\mu_0)$. The stationarity condition

$$\frac{\partial}{\partial \mu_0} (\Omega_{\text{bg}}(\mu_0) | (H^L - \mu N^L) \Omega_{\text{bg}}(\mu_0)) = 0$$

gives

$$\mu_0 = \mu - \kappa \mu_1 + o(\kappa). \quad (1.14)$$

The resulting term $-\kappa \mu_1 N^>$ is essential for the cancellation of the infrared singularities in the perturbative correction to the dispersion relation.

- (4) We decompose the grand-canonical Hamiltonian, obtaining a formula similar to (1.5), where now the terms $R_{4,i}^L$ and $H_{3,x}^L$ are absent. We compute the shift—the computations are almost identical as in the canonical setting, leading to

$$\delta^{\text{gcan}}(\mathbf{k}, \mu, \hat{v}) := \delta_4^{\text{gcan}}(\mathbf{k}, \mu, \hat{v}) + \delta_3^{\text{gcan}}(\mathbf{k}, \mu, \hat{v}). \quad (1.15)$$

- (5) Finally, we adjust μ_0 to reproduce the prescribed density. At the order considered here, the density relation is

$$n = n_{\text{bg}} = n_{0,\text{bg}} + n_{\text{bg}}^>, \quad \nu = \kappa \hat{v}(\mathbf{0})n = \mu_0 + \kappa \hat{v}(\mathbf{0})n_{\text{bg}}^>.$$

Thus we have a relationship between ν , μ_0 and μ . After expressing both the Bogoliubov dispersion and its perturbative correction in terms of ν , the coefficient c_{gcan} agrees with c_{can} through order κ .

The second main result of our paper yields the phonon velocity correction in the grand-canonical setting:

Theorem 1.2. *In the grand-canonical formalism, the dispersion relation computed up to the order κ has the following behaviour:*

$$\begin{aligned} \omega_{\text{bg}}(\mathbf{k}, \mu_0) + \kappa \delta_4^{\text{gcan}}(\mathbf{k}, \mu, \hat{v}) + \kappa \delta_3^{\text{gcan}}(\mathbf{k}, \mu, \hat{v}) \\ = |\mathbf{k}| c_{\text{gcan}}(\mu, \hat{v}) \left(1 + o\left(\left(|\mathbf{k}| \mu^{-\frac{1}{2}}\right)^0\right) \right) \text{ as } \frac{|\mathbf{k}|}{\sqrt{\mu}} \rightarrow 0, \end{aligned} \quad (1.16)$$

where μ_0 is computed from μ as in (1.14) and

$$c_{\text{gcan}}(\mu, \hat{v}) = \sqrt{\mu_0} - \frac{\kappa \hat{v}(\mathbf{0}) \sqrt{\mu_0}}{16} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{\hat{v}(\mathbf{p})^2}{\hat{v}(\mathbf{0})^2} \frac{|\mathbf{p}|^4}{\omega_{\text{bg}}(\mathbf{p}, \mu_0)^3} \quad (1.17)$$

$$+ \frac{\kappa \hat{v}(\mathbf{0})}{4\sqrt{\mu_0}} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{\left(\frac{|\mathbf{p}|^2}{2} + \frac{\hat{v}(\mathbf{p})}{\hat{v}(\mathbf{0})} \mu_0 - \omega_{\text{bg}}(\mathbf{p}, \mu_0)\right)}{\omega_{\text{bg}}(\mathbf{p}, \mu_0)}. \quad (1.18)$$

Let us stress that the grand-canonical and canonical pictures agree up to second order perturbation theory in $\sqrt{\kappa}$ for the phonon velocity:

$$c_{\text{can}}(\nu, \hat{v}) = c_{\text{gcan}}(\mu, \hat{v})(1 + o(\kappa)). \quad (1.19)$$

Note that dispersion relation is a much finer object than extensive quantities, for which the usual statement of the equivalence of the two pictures is usually stated.

Let us remark that when perturbation computations are performed in a naive way, then one obtains infrared-divergent terms of the form $\frac{1}{|\mathbf{k}|}$ in the small-momentum limit $\frac{|\mathbf{k}|}{\sqrt{\nu}} \rightarrow 0$. This was noted e.g. in [21, 23], where the imaginary part of the shift was computed, which does not require such a careful treatment. This point was already discussed in some of the older physics literature [1, 2, 27, 35], where it was argued that interacting particles always exhibit a depletion of the condensate number, so that the substitution $N_{\mathbf{0}} \simeq 1N$ is not justified and one omits many important contributions arising from disconnected diagrams [35].

In our computations the shifts δ_3^{can} and δ_4^{can} , as well as δ_3^{gcan} and δ_4^{gcan} , are given by relatively complicated expressions. Besides, all of them are infrared singular: they behave like $O(k^{-1})$. This infrared blowup cancels when we add them to obtain δ^{can} and δ^{gcan} . Besides, the formulas miraculously simplify, especially in the canonical approach.

In our paper we use the weak coupling approximation, because it has a rather clean mathematical meaning. Unfortunately, the weak coupling regime is usually a poor approximation of realistic quantum systems—e.g. their interactions typically possess a hard core. For alkali gases it is more appropriate to use the *dilute limit*. This is one of the reasons why this limit is more commonly used by both the

theoretical and mathematical physics communities. Note, however, that the dilute limit is not a good approximation for liquid helium. In particular, it does not lead to maxons and rotons, which can be visible in the weak coupling approximation.

The dilute limit is controlled by the density of the gas n and the *scattering length* $\mathfrak{a}$, where we take $n\mathfrak{a}^3 \rightarrow 0$. For weak coupling, $\mathfrak{a}$ can be expanded in a perturbative series $\sum_{n \geq 0} \mathfrak{a}_n \kappa^{n+1}$, where e.g. $\mathfrak{a}_0 = \frac{\hat{v}(\mathbf{0})}{4\pi}$. Using this we can recast the coefficient c_{can} in a rather elegant form:

Corollary 1.3. *In the limit $\kappa \rightarrow 0$, $\nu = \kappa n \hat{v}(\mathbf{0}) \rightarrow 0$ the following estimate holds*

$$c_{\text{can}}(\nu, \hat{v}) = \sqrt{4\pi n_{0,\text{bg}} \mathfrak{a}} \left(1 + \frac{7}{6\pi^2} \sqrt{n_{0,\text{bg}} (4\pi \mathfrak{a})^3} \right) (1 + o(\kappa^0)), \text{ as } \kappa \rightarrow 0. \quad (1.20)$$

Clearly, since $c_{\text{can}} = c_{\text{gcan}}(1 + o(\kappa))$, an analogous corollary holds also in the grand-canonical framework. The main term in Equation (1.20) is universal in the potential and recovers the well known expression first computed by Beliaev [2].

There exists a large mathematical literature about the Bose gas. A rigorous justification of Bogoliubov's c -number substitution in the mean-field setting was first established by Seiringer [50]; see also [39, 22, 32, 46] for related developments. Corresponding results in the Gross–Pitaevskii regime were subsequently obtained in [6, 11, 47], and extensions beyond this regime appear in [10]. Many of the results concerning the bottom of the excitation spectrum beyond Bogoliubov theory have recently been achieved in the Gross–Pitaevskii regime [16]. In the mean-field regime, analogous analysis was carried out in [9, 44]. However, almost none of the rigorous results concern quasiparticle dispersion relations. This is not very surprising, since the concept of quasiparticle dispersion relation is mathematically difficult.

On the other hand, quasiparticles are physically important and visible in experiments. Therefore, in the theoretical physics literature, they were considered very early. The series of two remarkable papers by Beliaev [1, 2], who computed the shift of the dispersion relation at zero temperature in the dilute limit, dates back to 1958. Beliaev used the Green functions approach in the canonical setting, organizing his computations with the help of diagrams (which were akin to, but different from, Feynman diagrams). The grand-canonical approach has been introduced to the Bose gas by Hugenholtz and Pines [35, Sec. 7]. Beliaev's result was rederived with similar methods in later papers [51, 43].

There exist a few papers that assume from the beginning that the interaction is given by a hard core potential and apply a method of *non-Hermitian Hamiltonians* [38, 42, 43, 54]. In this setting, Mohling and Sirlin [43] decomposed the interaction into *diagonal* and *off-diagonal* parts, separating terms involving occupation numbers from those involving creation and annihilation operators individually.

One can try to compute the energy-momentum spectrum of Bose gas using variational methods and quasifree states. This was first done by Arnowitt-Girardeau [29]; see also [52, 17]. Unfortunately, this method yields an (unphysical) gap in the spectrum; in particular, in this method it is difficult to determine the velocity of low momentum phonons.

As we mentioned above, quasiparticles are expected to be visible only in thermodynamic limit. Unfortunately, this limit is difficult to control rigorously. An attempt to study the energy-momentum spectrum in simultaneous large volume and mean field limit was made in [22]. The imaginary part of dispersion relation in the weak coupling regime was studied in [21], rederiving essentially the result of Beliaev. These results concerned zero temperature. Positive temperatures pose additional conceptual difficulties. The results of [21] were extended to low positive temperatures by the authors of this manuscript in [23] with the help of the Araki–Woods representations.

The Bose gas was also studied using methods of constructive quantum field theory. In [3, 4], Benfatto combined a functional integral representation with renormalization group methods to study Green functions of a weakly interacting three-dimensional Bose gas at zero temperature; see also [5]. At the level of renormalized perturbation theory, his analysis recovered the phonon behaviour of the infrared two-point function, with the sound velocity whose leading term agreed with the Bogoliubov prediction. The formulation was grand-canonical: the condensate density was prescribed, while the chemical potential was adjusted through a renormalization condition related to the Hugenholtz–Pines identity [35]. In our notation, the identification $\mu = \mu_0 = \nu$ therefore applied only at the leading order; the corrections to these parameters must be distinguished beyond the Bogoliubov approximation.

A different strategy, based on techniques from quantum field theory, was developed by Galanda and Pinamonti in [25, 26], where equilibrium states for an interacting Bose gas are constructed via a suitable resummation of perturbative expansions.

C^* -algebraic methods have been proposed for an interacting Bose gas by Buchholz and collaborators in [14, 15]. It is conceivable that the results about dispersion relations that we present in our paper could be recast in a more elegant form using those ideas.

We now describe the organization of the paper. Section 2 is devoted to the derivation of the effective Hamiltonian used to compute the shift of the dispersion relation in the canonical formalism. Section 3 contains the computation of this shift, which consists of the Hellmann–Feynman and the Fermi golden rule terms. We also compute the phonon velocity and compare it with the result of Beliaev for the dilute limit. In this section computations are done in a formal way—their rigorous justification is deferred to Section 6. Section 4 proposes a different derivation of the shift, based on the grand-canonical formalism. Section 5 deals with the estimates of the unphysical “tail terms” which appear in the effective Hamiltonian in the canonical setting. In Appendix A we recall the definition of the scattering length.

2. IMPROVED C -NUMBER SUBSTITUTION IN CANONICAL APPROACH

In this section we will discuss how to reorganize the Bose gas Hamiltonian in a way useful for calculations at a positive density. This derivation is based on the canonical formalism (large but fixed number of particles). There is a second possible derivation which uses the grand-canonical formalism (variable number of particles and fixed chemical potential). It is contained in Section 4, and it expands the derivation described e.g. in [21].

As a result of our derivation, the full Hamiltonian at a fixed density will be the sum of several terms. The main term will be the Bogoliubov Hamiltonian depending on the parameter ν . This parameter is closely related to the chemical potential, and proportional to the density of particles. The whole construction is a refined version of what is usually called the “ c -number substitution”.

We will first discuss the first-principles N -body Hamiltonian, in Subsections 2.1 and 2.2. Then we partition it in various terms in 2.3–2.12. At the end we obtain what can be called an “improved c -number substitution Hamiltonian”.

2.1. The first-principles Hamiltonian in infinite volume. As long as possible, we will keep the dimension d of our system arbitrary. However, we are mostly interested in $d = 3$, which we will assume in our concrete computations.

The following are our basic hypotheses on the interaction potential v for the preliminary analysis of the Hamiltonian. These hypotheses are sufficient to ensure

the possibility of introducing the usual Bogoliubov rotation and writing the *first-principle Hamiltonian* as the *improved c-number substitution Hamiltonian*.

Assumption 2.1. (Preliminary hypotheses on the potential) Assume there exists some $\nu_0 \in]0, +\infty[$ such that the potential satisfies the following assumptions for all the possible choices of $\nu \in]0, \nu_0]$

$$v \in L^1(\mathbb{R}^d, d\mathbf{x}), \quad v(\mathbf{x}) \in \mathbb{R}; \quad (2.1a)$$

$$\hat{v} \in L^2(\mathbb{R}^d, d\mathbf{k}); \quad (2.1b)$$

$$\hat{v}(\mathbf{0}) > 0, \quad \hat{v}(\mathbf{k}) > -\hat{v}(\mathbf{0}) \frac{|\mathbf{k}|^2}{4\nu}; \quad (2.1c)$$

$$v(\mathbf{x}) = v(-\mathbf{x}), \text{ which implies } \hat{v}(\mathbf{k}) = \hat{v}(-\mathbf{k}). \quad (2.1d)$$

Remark 2.2. In the following, when assuming the validity of Assumption 2.1 for a certain $\nu_0 > 0$, we will **always take** $\nu \in]0, \nu_0]$ without any further mention of this fact.

We would like to study N bosons interacting with potential v . In an infinite volume their Hamiltonian and momentum are

$$H_N := -\frac{1}{2} \sum_{i=1}^N \Delta_i + \sum_{1 \leq i < j \leq N} v(\mathbf{x}_i - \mathbf{x}_j), \quad (2.2)$$

$$\mathbf{P}_N := -i \sum_{i=1}^N \nabla_i. \quad (2.3)$$

They act on the Hilbert space

$$L_s^2((\mathbb{R}^d)^N) = \otimes_s^N L^2(\mathbb{R}^d). \quad (2.4)$$

One can also consider an arbitrary number of particles, using the formalism of second quantization. Then the Hamiltonian, number operator and the momentum are

$$H := \bigoplus_{N=0}^{\infty} H_N \quad (2.5)$$

$$= \int d\mathbf{x} \, a_{\mathbf{x}}^* \left(-\frac{1}{2} \Delta_{\mathbf{x}} \right) a_{\mathbf{x}} + \frac{1}{2} \int \int d\mathbf{x} d\mathbf{y} \, a_{\mathbf{x}}^* a_{\mathbf{y}}^* v(\mathbf{x} - \mathbf{y}) a_{\mathbf{y}} a_{\mathbf{x}}, \quad (2.6)$$

$$N := \bigoplus_{N=0}^{\infty} N = \int d\mathbf{x} \, a_{\mathbf{x}}^* a_{\mathbf{x}}, \quad (2.7)$$

$$\mathbf{P} := \bigoplus_{N=0}^{\infty} \mathbf{P}_N = -i \int d\mathbf{x} \, a_{\mathbf{x}}^* \nabla_{\mathbf{x}} a_{\mathbf{x}}, \quad (2.8)$$

acting on the Fock space $\Gamma_s(L^2(\mathbb{R}^d)) = \bigoplus_{N=0}^{\infty} L_s^2((\mathbb{R}^d)^N)$. (Note that we abuse the notation using the same letter N for the number operator and its eigenvalue.) Then, it will always be possible to recover the N -body Hamiltonian by projecting on $\otimes_s^N L^2(\mathbb{R}^d)$. One can rewrite these expressions in the momentum representation:

$$H = \int d\mathbf{p} \, \frac{1}{2} |\mathbf{p}|^2 a_{\mathbf{p}}^* a_{\mathbf{p}} + \frac{1}{2(2\pi)^d} \int d\mathbf{p} d\mathbf{q} d\mathbf{k} \, \hat{v}(\mathbf{k}) a_{\mathbf{p}+\mathbf{k}}^* a_{\mathbf{q}-\mathbf{k}}^* a_{\mathbf{q}} a_{\mathbf{p}}, \quad (2.9)$$

$$N = \int d\mathbf{p} \, a_{\mathbf{p}}^* a_{\mathbf{p}},$$

$$\mathbf{P} = \int d\mathbf{p} \, \mathbf{p} a_{\mathbf{p}}^* a_{\mathbf{p}}.$$

Unfortunately, if we are interested in a positive density of particles, these Hamiltonians are not very useful.

Some quantum systems in infinite volume can be elegantly described with the help of the C^* -algebraic approach. This works well for spin systems, sometimes also for fermionic systems: One introduces a C^* -algebra describing quasilocal observables equipped with a dynamics given by a family of automorphisms. Then one tries to consider a state on this C^* -algebra which describes a positive density of particles. Unfortunately, for interacting bosons this is problematic, because physically motivated dynamics often do not preserve our C^* -algebra. Maybe, this approach can be implemented using C^* -algebras introduced by Buchholz [14, 15]. Anyway, we will proceed with the traditional method: We will approximate the infinite volume system with a sequence of finite volume systems with periodic boundary conditions.

2.2. The first-principles Hamiltonian in finite volume. Let us consider a family of systems of finite size L . The space $\mathbb{R}^d$ is replaced with the d -dimensional cubic box of side length L , that is,

$$\Lambda_L = \left] -\frac{L}{2}, \frac{L}{2} \right]^d.$$

In order to preserve the translation invariance, Δ and ∇ are replaced with Δ^L and ∇^L with periodic boundary conditions acting on $L^2(\Lambda_L)$.

The potential is replaced by its *periodized* version

$$v^L(\mathbf{x}) := \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L} e^{i\mathbf{p}\mathbf{x}} \hat{v}(\mathbf{p}),$$

where $\Xi_L = \frac{2\pi}{L}\mathbb{Z}^d$ is the momentum lattice corresponding to Λ_L . Note that v^L is periodic with respect to the domain Λ_L and, for sufficiently decaying v , $v^L(\mathbf{x}) \rightarrow v(\mathbf{x})$ as $L \rightarrow \infty$.

Using Δ^L , ∇^L and v^L we can define the finite volume analogs of $H, N, \mathbf{P}$, denoted $H^L, N^L, \mathbf{P}^L$ acting on $\mathcal{H}^L := \Gamma_s(L^2(\Lambda_L))$, which will be the main objects of our study.

Using the identification $L^2(\Lambda_L) \simeq l^2(\Xi_L)$, we pass to the momentum representation. The Hamiltonian H^L , the number operator N^L and the momentum $\mathbf{P}^L$ are given by

$$H^L = \sum_{\mathbf{p} \in \Xi_L} \frac{1}{2} |\mathbf{p}|^2 a_{\mathbf{p}}^* a_{\mathbf{p}} + \frac{1}{2L^d} \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q} \in \Xi_L} \hat{v}(\mathbf{q}) a_{\mathbf{k}-\mathbf{q}}^* a_{\mathbf{p}+\mathbf{q}}^* a_{\mathbf{p}} a_{\mathbf{k}}, \quad (2.10)$$

$$N^L = \sum_{\mathbf{p} \in \Xi_L} a_{\mathbf{p}}^* a_{\mathbf{p}}, \quad (2.11)$$

$$\mathbf{P}^L = \sum_{\mathbf{p} \in \Xi_L} \mathbf{p} a_{\mathbf{p}}^* a_{\mathbf{p}}. \quad (2.12)$$

$n > 0$ is a fixed number called the *density of the gas*. For $L > 0$ we consider a family of systems with the total number of particles $N(L)$ such that

$$n(L) := \frac{N(L)}{L^d} = n(1 + o(L^0)), \quad \text{as } L \rightarrow +\infty. \quad (2.13)$$

Here $N(L)$ is the eigenvalue of the number operator N^L . Thus we assume that $N^L = N(L)$ throughout our paper, except in the section devoted to the grand-canonical approach.

Actually, instead of the density $n(L)$ and n we will usually prefer to use the parameter $\nu(L)$, resp. ν :

$$\nu(L) := n(L) \hat{v}(\mathbf{0}) = \frac{N(L)}{L^d} \hat{v}(\mathbf{0}), \quad \nu = n \hat{v}(\mathbf{0}), \quad (2.14)$$

whose meaning is roughly the chemical potential; see Section 4.

In what follows, we often suppress the label L and use ν , n for both the finite-volume quantities $\nu(L)$, $n(L)$ and their thermodynamic limit ν , n .

2.3. Diagonal interaction. Let us introduce the operators

$$N_{\mathbf{p}} := a_{\mathbf{p}}^* a_{\mathbf{p}}. \quad (2.15)$$

We begin by decomposing the interaction term into a *diagonal* and an *off-diagonal* component:

$$\frac{1}{2L^d} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{k}} \hat{v}(\mathbf{k}) a_{\mathbf{p}+\mathbf{k}}^* a_{\mathbf{q}-\mathbf{k}}^* a_{\mathbf{q}} a_{\mathbf{p}} = V_{\text{diag}} + V_{\text{off}}, \quad (2.16)$$

where the diagonal component is given by

$$V_{\text{diag}} = \frac{1}{2L^d} \sum_{\mathbf{p}, \mathbf{q}} N_{\mathbf{p}} N_{\mathbf{q}} (\hat{v}(\mathbf{0}) + \hat{v}(\mathbf{q} - \mathbf{p})) - \frac{\hat{v}(\mathbf{0})}{2L^d} \sum_{\mathbf{p}} N_{\mathbf{p}} (N_{\mathbf{p}} + 1). \quad (2.17)$$

We proceed now to separate the condensate components, i.e. the zero momentum mode, in the diagonal component. Introducing the momentum lattice outside of the condensate $\Xi_L^{\geq} := \Xi_L \setminus \{\mathbf{0}\}$, we have

$$\begin{aligned} V_{\text{diag}} = & \frac{1}{2L^d} \left(\hat{v}(\mathbf{0}) N_{\mathbf{0}} (N_{\mathbf{0}} - 1) + 2N_{\mathbf{0}} \sum_{\mathbf{p} \in \Xi_L^{\geq}} N_{\mathbf{p}} (\hat{v}(\mathbf{0}) + \hat{v}(\mathbf{p})) \right. \\ & \left. + \sum_{\mathbf{p}, \mathbf{q} \in \Xi_L^{\geq}} N_{\mathbf{p}} N_{\mathbf{q}} (\hat{v}(\mathbf{0}) + \hat{v}(\mathbf{p} - \mathbf{q})) - \hat{v}(\mathbf{0}) \sum_{\mathbf{p} \in \Xi_L^{\geq}} N_{\mathbf{p}} (N_{\mathbf{p}} + 1) \right). \end{aligned} \quad (2.18)$$

In the canonical formalism we can directly substitute the equality

$$N_{\mathbf{0}} = N - \sum_{\mathbf{k} \in \Xi_L^{\geq}} N_{\mathbf{k}} \quad (2.19)$$

inside this expression, obtaining

$$\begin{aligned} V_{\text{diag}} = & \frac{1}{2L^d} \left(\hat{v}(\mathbf{0}) N(N - 1) - (2N - 1) \hat{v}(\mathbf{0}) \sum_{\mathbf{p} \in \Xi_L^{\geq}} N_{\mathbf{p}} + \hat{v}(\mathbf{0}) \sum_{\mathbf{p}, \mathbf{q} \in \Xi_L^{\geq}} N_{\mathbf{q}} N_{\mathbf{p}} \right. \\ & + 2N \sum_{\mathbf{p} \in \Xi_L^{\geq}} N_{\mathbf{p}} (\hat{v}(\mathbf{0}) + \hat{v}(\mathbf{p})) - 2 \sum_{\mathbf{p}, \mathbf{q} \in \Xi_L^{\geq}} N_{\mathbf{q}} N_{\mathbf{p}} (\hat{v}(\mathbf{0}) + \hat{v}(\mathbf{p})) \\ & \left. + \sum_{\mathbf{p}, \mathbf{q} \in \Xi_L^{\geq}} N_{\mathbf{p}} N_{\mathbf{q}} (\hat{v}(\mathbf{0}) + \hat{v}(\mathbf{p} - \mathbf{q})) - \hat{v}(\mathbf{0}) \sum_{\mathbf{p} \in \Xi_L^{\geq}} N_{\mathbf{p}} (N_{\mathbf{p}} + 1) \right) \end{aligned} \quad (2.20)$$

$$\begin{aligned} = & \frac{1}{2L^d} \left(\hat{v}(\mathbf{0}) N(N - 1) + 2N \sum_{\mathbf{p} \in \Xi_L^{\geq}} N_{\mathbf{p}} \hat{v}(\mathbf{p}) \right. \\ & \left. - \sum_{\mathbf{p}, \mathbf{q} \in \Xi_L^{\geq}} N_{\mathbf{q}} N_{\mathbf{p}} (2\hat{v}(\mathbf{p}) - \hat{v}(\mathbf{p} - \mathbf{q})) \right) - \frac{\hat{v}(\mathbf{0})}{2L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} N_{\mathbf{p}}^2. \end{aligned} \quad (2.21)$$

In thermodynamic limit $L \rightarrow +\infty$, we expect that the most relevant terms are the ones containing either a product of N with a series over momenta, or the product of two such series. Thus the last term in line (2.21) is expected to "vanish" in thermodynamic limit.

2.4. Extended space. Recall that $\Xi_L^> := \Xi_L \setminus \{\mathbf{0}\}$. By the exponential property of Fock spaces we have the identification

$$\Gamma_s(l^2(\Xi_L)) \simeq \Gamma_s(\mathbb{C}) \otimes \Gamma_s(l^2(\Xi_L^>)), \quad (2.22)$$

where the first factor describes the *zeroth mode* treated as the *condensate* and the second “excitations outside of the condensate”. Let us embed the space of zero-modes $\Gamma_s(\mathbb{C}) = l^2(\{0, 1, \dots\})$ in a larger space $l^2(\mathbb{Z})$. Thus we obtain the *extended Hilbert space*

$$\mathcal{H}^{\text{ext}} := l^2(\mathbb{Z}) \otimes \Gamma_s(l^2(\Xi_L^>)), \quad (2.23)$$

where in the notation we suppress the dependence on L .

The physical space (2.22) is spanned by vectors of the form $|N_{\mathbf{0}}\rangle \otimes \Psi^>$, where $|N_{\mathbf{0}}\rangle$ represents $N_{\mathbf{0}}$ zero-modes ($N_{\mathbf{0}} \geq 0$) and $\Psi^>$ represents a vector outside the zero-mode. The space (2.23) is also spanned by vectors of this form, where now the relation $N_{\mathbf{0}} \geq 0$ is not imposed.

On $\mathcal{H}^{\text{ext}}$ we have a self-adjoint operator $N_{\mathbf{0}}^{\text{ext}}$ such that

$$N_{\mathbf{0}}^{\text{ext}}|N_{\mathbf{0}}\rangle \otimes \Psi^> = N_{\mathbf{0}}|N_{\mathbf{0}}\rangle \otimes \Psi^>. \quad (2.24)$$

Its spectrum equals $\mathbb{Z}$. We also define

$$N^> := \sum_{\mathbf{p} \in \Xi_L^>} N_{\mathbf{p}}, \quad (2.25)$$

$$N^{\text{ext}} := N^> + N_{\mathbf{0}}^{\text{ext}}. \quad (2.26)$$

If $N \in \mathbb{Z}$, we will write $\mathcal{H}_N^{\text{ext}}$ for the subspace of $\mathcal{H}^{\text{ext}}$ corresponding to $N^{\text{ext}} = N$.

For $t \in \mathbb{R}$ we will use the standard notation $|t|_+ := \max(0, t)$. We will write

$$N_{\mathbf{0}}^+ := \max(0, N_{\mathbf{0}}^{\text{ext}}) = |N_{\mathbf{0}}^{\text{ext}}|_+. \quad (2.27)$$

Clearly

$$N_{\mathbf{0}}^+ \upharpoonright \Gamma_s(l^2(\Xi_L)) = N_{\mathbf{0}}^{\text{ext}} \upharpoonright \Gamma_s(l^2(\Xi_L)) = N_{\mathbf{0}}, \quad \Gamma_s(l^2(\Xi_L)) = \text{Ran} \mathbb{1}_{[0, \infty[}(N_{\mathbf{0}}^{\text{ext}}).$$

We have also a unitary operator

$$U|N_{\mathbf{0}}\rangle \otimes \Psi^> = |N_{\mathbf{0}} - 1\rangle \otimes \Psi^>. \quad (2.28)$$

Notice that both U and $U^\dagger$ commute with both $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^*$ with $\mathbf{p} \neq \mathbf{0}$. We now define for $\mathbf{p} \neq \mathbf{0}$ the following operator on $\mathcal{H}^{\text{ext}}$:

$$\tilde{a}_{\mathbf{p}} := a_{\mathbf{p}} U^*. \quad (2.29)$$

Operators $\tilde{a}_{\mathbf{p}}$ and $\tilde{a}_{\mathbf{q}}^*$ satisfy the same CCR as $a_{\mathbf{p}}$ and $a_{\mathbf{q}}^*$. However, they do not commute with $N_{\mathbf{0}}^{\text{ext}}$:

$$(N_{\mathbf{0}}^{\text{ext}} - 1)\tilde{a}_{\mathbf{p}} = \tilde{a}_{\mathbf{p}} N_{\mathbf{0}}^{\text{ext}}. \quad (2.30)$$

We extend $a_{\mathbf{0}}$ to $\mathcal{H}^{\text{ext}}$ by setting $a_{\mathbf{0}}|N_{\mathbf{0}}\rangle = 0$, $N_{\mathbf{0}} \leq 0$, and consequently $a_{\mathbf{0}}^*|N_{\mathbf{0}}\rangle = 0$, $N_{\mathbf{0}} < 0$. Note that

$$U^* a_{\mathbf{0}} = \sqrt{N_{\mathbf{0}}^+}, \quad (2.31)$$

$$U^* N_{\mathbf{0}}^{\text{ext}} U = N_{\mathbf{0}}^{\text{ext}} - 1, \quad U^* N^{\text{ext}} U = N^{\text{ext}} - 1. \quad (2.32)$$

Let H^{ext} be the operator on $\mathcal{H}^{\text{ext}}$ given by the same expression as (2.10), where now the operators $a_{\mathbf{0}}$, $a_{\mathbf{0}}^*$ have the new meaning.

H^{ext} preserves the direct sum decomposition

$$\mathcal{H}^{\text{ext}} = \dots \oplus \text{Ran} \mathbb{1}_{\{-2\}}(N_{\mathbf{0}}^{\text{ext}}) \oplus \text{Ran} \mathbb{1}_{\{-1\}}(N_{\mathbf{0}}^{\text{ext}}) \oplus \Gamma_s(l^2(\Xi_L)). \quad (2.33)$$

On the last summand H^{ext} coincides with the physical Hamiltonian H^L .

It is easy to see that all terms in H^{ext} commute with N^{ext} .

2.5. Off-diagonal interaction. The off-diagonal part of the interaction is

$$V_{\text{off}} = \frac{1}{2L^d} \sum_{\mathbf{k}, \mathbf{q}-\mathbf{p}-\mathbf{k} \in \Xi_L^{\geq}} \hat{v}(\mathbf{k}) a_{\mathbf{p}+\mathbf{k}}^* a_{\mathbf{q}-\mathbf{k}}^* a_{\mathbf{q}} a_{\mathbf{p}}. \quad (2.34)$$

Using the shorthand

$$A_{\mathbf{p}} = A_{-\mathbf{p}} = a_{\mathbf{p}} a_{-\mathbf{p}}, \quad (2.35)$$

$$A_{\mathbf{p}}^* = A_{-\mathbf{p}}^* = a_{-\mathbf{p}}^* a_{\mathbf{p}}^*, \quad (2.36)$$

the off diagonal part can be written as

$$V_{\text{off}} = \frac{1}{2L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} \hat{v}(\mathbf{p}) (A_0^* A_{\mathbf{p}} + \text{h.c.}) \quad (2.37)$$

$$+ \frac{1}{L^d} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{p}+\mathbf{q} \in \Xi_L^{\geq}} \hat{v}(\mathbf{p}) (a_0^* a_{\mathbf{p}+\mathbf{q}}^* a_{\mathbf{p}} a_{\mathbf{q}} + \text{h.c.}) \quad (2.38)$$

$$+ \frac{1}{2L^d} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{p} \pm \mathbf{q} \in \Xi_L^{\geq}} \hat{v}(\mathbf{p}-\mathbf{q}) A_{\mathbf{p}}^* A_{\mathbf{q}} \quad (2.39)$$

$$+ \frac{1}{2L^d} \sum_{\substack{\mathbf{p}, \mathbf{q}, \mathbf{k}, \mathbf{p}+\mathbf{q}, \\ \mathbf{p}+\mathbf{k}, \mathbf{q}-\mathbf{k}, \mathbf{p}+\mathbf{k}-\mathbf{q} \in \Xi_L^{\geq}}} \hat{v}(\mathbf{k}) a_{\mathbf{p}+\mathbf{k}}^* a_{\mathbf{q}-\mathbf{k}}^* a_{\mathbf{q}} a_{\mathbf{p}}. \quad (2.40)$$

Note new conditions under the sum in (2.40): $\mathbf{p}+\mathbf{k}$, $\mathbf{q}-\mathbf{k}$, $\mathbf{p}$, $\mathbf{q}$ should be nonzero because the zero components are included in (2.37) and (2.38); the $\mathbf{p}+\mathbf{q}=0$ terms are included in (2.37) and (2.39). The conditions $\mathbf{p}, \mathbf{q} \neq 0$ under (2.39) exclude terms which were in (2.37) and the N_0 terms of the diagonal part.

Let us use the extended space, setting

$$\tilde{A}_{\mathbf{p}} := \tilde{a}_{\mathbf{p}} \tilde{a}_{-\mathbf{p}} = U^{*2} A_{\mathbf{p}}, \quad \mathbf{p} \in \Xi_L^{\geq}; \quad (2.41)$$

$$\text{and using } \sqrt{N_0^+(N_0^+ - 1)} = U^{*2} a_0^2 = U^{*2} A_0. \quad (2.42)$$

In the extended space the off-diagonal interaction becomes

$$V_{\text{off}} = \frac{1}{2L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} \hat{v}(\mathbf{p}) \sqrt{N_0^+(N_0^+ - 1)} \tilde{A}_{\mathbf{p}} + \text{h.c.} \quad (2.43)$$

$$+ \frac{1}{L^d} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{p}+\mathbf{q} \in \Xi_L^{\geq}} \hat{v}(\mathbf{p}) \sqrt{N_0^+} \tilde{a}_{\mathbf{p}+\mathbf{q}}^* \tilde{a}_{\mathbf{p}} \tilde{a}_{\mathbf{q}} + \text{h.c.} \quad (2.44)$$

$$+ \frac{1}{2L^d} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{p} \pm \mathbf{q} \in \Xi_L^{\geq}} \hat{v}(\mathbf{p}-\mathbf{q}) \tilde{A}_{\mathbf{p}}^* \tilde{A}_{\mathbf{q}} \quad (2.45)$$

$$+ \frac{1}{2L^d} \sum_{\substack{\mathbf{p}, \mathbf{q}, \mathbf{k}, \mathbf{p}+\mathbf{q}, \\ \mathbf{p}+\mathbf{k}, \mathbf{q}-\mathbf{k}, \mathbf{p}+\mathbf{k}-\mathbf{q} \in \Xi_L^{\geq}}} \hat{v}(\mathbf{k}) \tilde{a}_{\mathbf{p}+\mathbf{k}}^* \tilde{a}_{\mathbf{q}-\mathbf{k}}^* \tilde{a}_{\mathbf{q}} \tilde{a}_{\mathbf{p}}. \quad (2.45)$$

We rewrite the first two lines of the off-diagonal potential as

$$(2.43) = \frac{1}{2L^d} \left(\sqrt{N_0^+(N_0^+ - 1)} - \sqrt{N_0^{\text{ext}}(N_0^{\text{ext}} - 1)} \right) \sum_{\mathbf{p} \in \Xi_L^{\geq}} \hat{v}(\mathbf{p}) \tilde{A}_{\mathbf{p}} + \text{h.c.} \quad (2.46)$$

$$+ \frac{1}{2L^d} \left(\sqrt{N_0^{\text{ext}}(N_0^{\text{ext}} - 1)} - N_0^{\text{ext}} + \frac{1}{2} \right) \sum_{\mathbf{p} \in \Xi_L^{\geq}} \hat{v}(\mathbf{p}) \tilde{A}_{\mathbf{p}} + \text{h.c.} \quad (2.47)$$

$$+ \frac{1}{2L^d} \left(N_0^{\text{ext}} - \frac{1}{2} \right) \sum_{\mathbf{p} \in \Xi_L^{\geq}} \hat{v}(\mathbf{p}) \tilde{A}_{\mathbf{p}} + \text{h.c.} \quad (2.48)$$

and

$$(2.44) = \frac{1}{L^d} \left(\sqrt{N_0^+} - \sqrt{N} \right) \sum_{\mathbf{p}, \mathbf{q}, \mathbf{p}+\mathbf{q} \in \Xi_L^>} \hat{v}(\mathbf{p}) \tilde{a}_{\mathbf{p}+\mathbf{q}}^* \tilde{a}_{\mathbf{p}} \tilde{a}_{\mathbf{q}} + \text{h.c.} \quad (2.49)$$

$$+ \frac{1}{L^d} \sqrt{N} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{p}+\mathbf{q} \in \Xi_L^>} \hat{v}(\mathbf{p}) \tilde{a}_{\mathbf{p}+\mathbf{q}}^* \tilde{a}_{\mathbf{p}} \tilde{a}_{\mathbf{q}} + \text{h.c.} \quad (2.50)$$

We will see that (2.46), (2.47) are negligible in thermodynamic limit. Introducing the parameter ν (2.14) we further transform

$$(2.48) = \frac{\nu}{2\hat{v}(\mathbf{0})} \sum_{\mathbf{p} \in \Xi_L^>} \hat{v}(\mathbf{p}) \tilde{A}_{\mathbf{p}}^* + \text{h.c.} \quad (2.51)$$

$$- \frac{1}{4L^d} \sum_{\mathbf{p} \in \Xi_L^>} \hat{v}(\mathbf{p}) \tilde{A}_{\mathbf{p}}^* + \text{h.c.}$$

$$- \frac{1}{2L^d} \sum_{\mathbf{q} \in \Xi_L^>} N_{\mathbf{q}} \sum_{\mathbf{p} \in \Xi_L^>} \hat{v}(\mathbf{p}) \tilde{A}_{\mathbf{p}}^* + \text{h.c.}$$

$$(2.49) = \frac{\sqrt{\nu}}{L^{\frac{d}{2}} \sqrt{\hat{v}(\mathbf{0})}} \left(\left| 1 - \frac{\hat{v}(\mathbf{0})}{\nu L^d} \sum_{\mathbf{p} \in \Xi_L^>} N_{\mathbf{p}} \right|_+^{\frac{1}{2}} - 1 \right) \sum_{\mathbf{p}, \mathbf{q}, \mathbf{p}+\mathbf{q} \in \Xi_L^>} \hat{v}(\mathbf{p}) \tilde{a}_{\mathbf{p}+\mathbf{q}}^* \tilde{a}_{\mathbf{p}} \tilde{a}_{\mathbf{q}} + \text{h.c.}$$

$$(2.50) = \frac{\sqrt{\nu}}{L^{\frac{d}{2}} \sqrt{\hat{v}(\mathbf{0})}} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{p}+\mathbf{q} \in \Xi_L^>} \hat{v}(\mathbf{p}) \tilde{a}_{\mathbf{p}+\mathbf{q}}^* \tilde{a}_{\mathbf{p}} \tilde{a}_{\mathbf{q}} + \text{h.c.} \quad (2.52)$$

From now on we will omit the tildes from the notation.

2.6. Full Hamiltonian. Let us include also a dimensionless coupling constant κ . We will rescale the potential as $\hat{v} \rightarrow \kappa \hat{v}$, with κ assumed *small but finite* throughout the computation. So, taking into account of this rescaling, $\nu(L) = \kappa n(L) \hat{v}(\mathbf{0})$.

The full interacting Hamiltonian in the extended space becomes

$$H^L = H_{\text{bg}}^L + \sqrt{\kappa} H_3^L + \kappa H_2^L + \kappa H_{4,2}^L + \kappa H_{4,1}^L + \kappa H_{4,3}^L + \kappa \sqrt{\kappa} H_{3,x}^L \quad (2.53)$$

$$+ \kappa R_{4,1}^L + \kappa R_{4,2}^L, \quad (2.54)$$

where we have introduced the Hamiltonian operators

$$\begin{aligned}
H_{\text{bg}}^L &:= \sum_{\mathbf{p} \in \Xi_L^{\geq}} \left(\frac{|\mathbf{p}|^2}{2} + \frac{\nu \hat{v}(\mathbf{p})}{\hat{v}(\mathbf{0})} \right) N_{\mathbf{p}} + \sum_{\mathbf{p} \in \Xi_L^{\geq}} \frac{\nu \hat{v}(\mathbf{p})}{2 \hat{v}(\mathbf{0})} (A_{\mathbf{p}}^* + \text{h.c.}) \\
H_2^L &:= -\frac{1}{4L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} \hat{v}(\mathbf{p}) A_{\mathbf{p}}^* + \text{h.c.} \\
H_3^L &:= \frac{1}{L^{\frac{d}{2}}} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{p}+\mathbf{q} \in \Xi_L^{\geq}} \sqrt{\frac{\nu}{\hat{v}(\mathbf{0})}} \hat{v}(\mathbf{p}) (a_{\mathbf{p}+\mathbf{q}}^* a_{\mathbf{p}} a_{\mathbf{q}} + \text{h.c.}) \\
H_{3,x}^L &:= \frac{\sqrt{\nu}}{L^{\frac{d}{2}} \sqrt{\hat{v}(\mathbf{0})} \kappa} \left(\left| 1 - \frac{\kappa \hat{v}(\mathbf{0})}{\nu L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} N_{\mathbf{p}} \right|_+^{\frac{1}{2}} - 1 \right) \sum_{\mathbf{p}, \mathbf{q}, \mathbf{p}+\mathbf{q} \in \Xi_L^{\geq}} \hat{v}(\mathbf{p}) a_{\mathbf{p}+\mathbf{q}}^* a_{\mathbf{p}} a_{\mathbf{q}} + \text{h.c.} \\
H_{4,2}^L &:= -\frac{1}{2L^d} \sum_{\mathbf{q} \in \Xi_L^{\geq}} N_{\mathbf{q}} \sum_{\mathbf{p} \in \Xi_L^{\geq}} N_{\mathbf{p}} (\hat{v}(\mathbf{p}) + \hat{v}(\mathbf{q}) - \hat{v}(\mathbf{p} - \mathbf{q})) \\
&\quad - \frac{1}{2L^d} \sum_{\mathbf{q} \in \Xi_L^{\geq}} N_{\mathbf{q}} \sum_{\mathbf{p} \in \Xi_L^{\geq}} \hat{v}(\mathbf{p}) A_{\mathbf{p}}^* + \text{h.c.} \\
&\quad + \frac{1}{2L^d} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{p} \pm \mathbf{q} \in \Xi_L^{\geq}} \hat{v}(\mathbf{p} - \mathbf{q}) A_{\mathbf{p}}^* A_{\mathbf{q}}, \\
H_{4,1}^L &:= -\frac{\hat{v}(\mathbf{0})}{2L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} N_{\mathbf{p}}^2 \\
H_{4,3}^L &:= \frac{1}{2L^d} \sum_{\substack{\mathbf{p}, \mathbf{q}, \mathbf{k}, \mathbf{p}+\mathbf{q}, \\ \mathbf{p}+\mathbf{k}, \mathbf{q}-\mathbf{k}, \mathbf{p}+\mathbf{k}-\mathbf{q} \in \Xi_L^{\geq}}} \hat{v}(\mathbf{k}) a_{\mathbf{p}+\mathbf{k}}^* a_{\mathbf{q}-\mathbf{k}}^* a_{\mathbf{q}} a_{\mathbf{p}},
\end{aligned}$$

and the "tail terms"

$$\begin{aligned}
R_{4,1}^L &:= \frac{1}{2L^d} \left(\sqrt{N_0^+ (N_0^+ - 1)} - \sqrt{N_0^{\text{ext}} (N_0^{\text{ext}} - 1)} \right) \sum_{\mathbf{p} \in \Xi_L^{\geq}} \hat{v}(\mathbf{p}) A_{\mathbf{p}} + \text{h.c.} \\
R_{4,2}^L &:= \frac{1}{2L^d} \left(\sqrt{N_0^{\text{ext}} (N_0^{\text{ext}} - 1)} - N_0^{\text{ext}} + \frac{1}{2} \right) \sum_{\mathbf{p} \in \Xi_L^{\geq}} \hat{v}(\mathbf{p}) A_{\mathbf{p}} + \text{h.c.}
\end{aligned}$$

2.7. Bogoliubov rotation. The Hamiltonian H_{bg}^L , as well as its thermodynamic limit H_{bg} , will be called the *Bogoliubov Hamiltonian* and they will be treated as the main part of the full Hamiltonian. It is purely quadratic, therefore it can be diagonalized using an appropriate *Bogoliubov transformation*.

In fact, consider $\theta = (\theta_{\mathbf{k}})_{\mathbf{k} \in \Xi_L^{\geq}}$, a sequence of real numbers. Set

$$c_{\mathbf{k}} := \cosh \theta_{\mathbf{k}}, \quad s_{\mathbf{k}} := \sinh \theta_{\mathbf{k}}. \quad (2.55)$$

For $\mathbf{k} \in \Xi_L^{\geq}$ we make the substitution

$$a_{\mathbf{k}}^* = c_{\mathbf{k}} b_{\mathbf{k}}^* - s_{\mathbf{k}} b_{-\mathbf{k}}, \quad a_{\mathbf{k}} = c_{\mathbf{k}} b_{\mathbf{k}} - s_{\mathbf{k}} b_{-\mathbf{k}}^*. \quad (2.56)$$

We choose the Bogoliubov rotation that kills double creators and annihilators, which amounts to choosing $s_{\mathbf{k}}$ as

$$s_{\mathbf{k}} = \text{sign}(\hat{v}(\mathbf{k})) \frac{1}{\sqrt{2}} \left(\left(1 - \left(\frac{\hat{v}(\mathbf{k}) \frac{\nu}{\hat{v}(\mathbf{0})}}{\frac{1}{2} \mathbf{k}^2 + \hat{v}(\mathbf{k}) \frac{\nu}{\hat{v}(\mathbf{0})}} \right)^2 \right)^{-1/2} - 1 \right)^{1/2}, \quad (2.57)$$

and $c_{\mathbf{k}} = \sqrt{1 + s_{\mathbf{k}}^2}$. Note that we have $s_{\mathbf{k}} = s_{-\mathbf{k}}$ and $c_{\mathbf{k}} = c_{-\mathbf{k}}$. Another useful relation between $c_{\mathbf{k}}$ and $s_{\mathbf{k}}$ is

$$s_{\mathbf{k}} = \frac{\hat{v}(\mathbf{k})\nu}{2\omega_{\text{bg}}(\mathbf{k})\hat{v}(\mathbf{0})} \frac{1}{c_{\mathbf{k}}}. \quad (2.58)$$

Equation (2.57) is well defined on account of Assumption (2.1c). Moreover, Assumption (2.1a) ensures that $\hat{v} \in C_0(\mathbb{R}^d)$. Then, we see that

$$s_{\mathbf{k}}^2 = O\left(\frac{\hat{v}(\mathbf{k})^2}{|\mathbf{k}|^4}\right) \text{ as } |\mathbf{k}| \rightarrow +\infty, \quad (2.59)$$

$$s_{\mathbf{k}}^2 = O\left(\frac{1}{|\mathbf{k}|}\right) \text{ as } |\mathbf{k}| \rightarrow 0, \quad (2.60)$$

and thus, $s_{\mathbf{k}} \in \ell^2(\Xi_L^>)$ for $d = 2, 3$. This last property continues to hold also in the flat Fourier limit $\hat{v}(\mathbf{k}) = \hat{v}(\mathbf{0})$. Moreover, $s_{\mathbf{k}}$ can be also defined for $\mathbf{k} \in \mathbb{R}^d$, and as a consequence of its decay properties at infinity we see that

$$\lim_{L \rightarrow +\infty} \frac{1}{L^d} \sum_{\mathbf{k} \in \Xi_L^>} s_{\mathbf{k}}^2 = \int \frac{d\mathbf{k}}{(2\pi)^d} s_{\mathbf{k}}^2 < +\infty. \quad (2.61)$$

After the diagonalization, we obtain

$$H_{\text{bg}}^L = E_{\text{bg}}^L + \sum_{\mathbf{k} \in \Xi_L^>} \omega_{\text{bg}}(\mathbf{k}) b_{\mathbf{k}}^* b_{\mathbf{k}}, \quad (2.62)$$

$$\mathbf{P}^L = \sum_{\mathbf{k} \in \Xi_L^>} \mathbf{k} b_{\mathbf{k}}^* b_{\mathbf{k}}, \quad (2.63)$$

where the quasiparticle excitation spectrum is defined as

$$\omega_{\text{bg}}(\mathbf{k}) = \sqrt{\frac{|\mathbf{k}|^4}{4} + \frac{\hat{v}(\mathbf{k})}{\hat{v}(\mathbf{0})} \nu |\mathbf{k}|^2}. \quad (2.64)$$

The ground state energy is

$$E_{\text{bg}}^L = - \sum_{\mathbf{k} \in \Xi_L^>} \frac{1}{2} \left(\frac{1}{2} \mathbf{k}^2 + \nu \frac{\hat{v}(\mathbf{k})}{\hat{v}(\mathbf{0})} - \omega_{\text{bg}}(\mathbf{k}) \right). \quad (2.65)$$

In the following instead of H_{bg}^L in Equation (2.62) we will use the centered version $H_{\text{bg}}^L - E_{\text{bg}}^L$. With some abuse of notation, this will be again denoted by H_{bg}^L .

It is convenient to introduce the operators

$$M_{\mathbf{p}} := b_{\mathbf{p}}^* b_{\mathbf{p}}, \quad (2.66)$$

$$B_{\mathbf{p}} = B_{-\mathbf{p}} := b_{\mathbf{p}} b_{-\mathbf{p}}, \quad (2.67)$$

$$B_{\mathbf{p}}^* = B_{-\mathbf{p}}^* := b_{-\mathbf{p}}^* b_{\mathbf{p}}^*. \quad (2.68)$$

When we express the full Hamiltonian H^L in the Bogoliubov operators, the following identities are useful:

$$N_{\mathbf{p}} = s_{\mathbf{p}}^2 + c_{\mathbf{p}}^2 M_{\mathbf{p}} + s_{\mathbf{p}}^2 M_{-\mathbf{p}} - s_{\mathbf{p}} c_{\mathbf{p}} (B_{\mathbf{p}} + B_{\mathbf{p}}^*), \quad (2.69)$$

$$A_{\mathbf{p}} = -s_{\mathbf{p}} c_{\mathbf{p}} - s_{\mathbf{p}} c_{\mathbf{p}} (M_{\mathbf{p}} + M_{-\mathbf{p}}) + c_{\mathbf{p}}^2 B_{\mathbf{p}} + s_{\mathbf{p}}^2 B_{\mathbf{p}}^*; \quad (2.70)$$

$$[N_{\mathbf{p}}, b_{\mathbf{k}}^*] = c_{\mathbf{p}}^2 b_{\mathbf{k}}^* \delta_{\mathbf{p}\mathbf{k}} + s_{\mathbf{p}}^2 b_{\mathbf{k}}^* \delta_{-\mathbf{p}\mathbf{k}} - s_{\mathbf{p}} c_{\mathbf{p}} b_{-\mathbf{k}} (\delta_{\mathbf{p}\mathbf{k}} + \delta_{-\mathbf{p}\mathbf{k}}), \quad (2.71)$$

$$[A_{\mathbf{p}}, b_{\mathbf{k}}^*] = -s_{\mathbf{p}} c_{\mathbf{p}} b_{\mathbf{k}}^* \delta_{\mathbf{p}\mathbf{k}} - s_{\mathbf{p}} c_{\mathbf{p}} b_{\mathbf{k}}^* \delta_{-\mathbf{p}\mathbf{k}} + c_{\mathbf{p}}^2 b_{-\mathbf{k}} (\delta_{\mathbf{p}\mathbf{k}} + \delta_{-\mathbf{p}\mathbf{k}}); \quad (2.72)$$

$$[b_{\mathbf{k}}, [N_{\mathbf{p}}, b_{\mathbf{k}}^*]] = c_{\mathbf{p}}^2 \delta_{\mathbf{p}\mathbf{k}} + s_{\mathbf{p}}^2 \delta_{-\mathbf{p}\mathbf{k}}, \quad (2.73)$$

$$[b_{\mathbf{k}}, [A_{\mathbf{p}}, b_{\mathbf{k}}^*]] = -s_{\mathbf{p}} c_{\mathbf{p}} (\delta_{\mathbf{p}\mathbf{k}} + \delta_{-\mathbf{p}\mathbf{k}}). \quad (2.74)$$

The ground state of the Bogoliubov Hamiltonian H_{bg}^L in the extended space in the N -particle sector is

$$\Omega_{\text{bg}} = U_\theta |N\rangle \otimes \Omega^> = \prod_{\mathbf{p} \in \Xi_L^>} \frac{1}{(\cosh \theta_{\mathbf{p}})^{\frac{1}{2}}} e^{-\frac{1}{2} \tanh \theta_{\mathbf{p}} a_{\mathbf{p}}^* a_{-\mathbf{p}}^*} |N\rangle \otimes \Omega^>, \quad (2.75)$$

where $\Omega^>$ is the Fock vacuum of $\Gamma_s(l^2(\Xi_L^>))$. It is important to remark that the operators $a_{\mathbf{k}}^* a_{-\mathbf{k}}^*$ act also on $|N\rangle$, by lowering its value by two. In particular, the state Ω_{bg} sticks out of the physical space by its tail. We can further exploit the symmetry under reflection to write

$$\Omega_{\text{bg}} = \prod_{\mathbf{p} \in \Xi_L^{>,+}} \frac{1}{\cosh \theta_{\mathbf{p}}} e^{-\tanh \theta_{\mathbf{p}} a_{\mathbf{p}}^* a_{-\mathbf{p}}^*} |N\rangle \otimes \Omega^>, \quad (2.76)$$

where we have added an additional constrain

$$\begin{aligned} \Xi_L^{>,+} = & \{\mathbf{p} \in \Xi_L^> | p_d > 0\} \cup \{\mathbf{p} \in \Xi_L^> | p_d = 0, p_{d-1} > 0\} \cup \dots \\ & \dots \cup \{\mathbf{p} \in \Xi_L^> | p_d = \dots = p_2 = 0, p_1 > 0\}. \end{aligned} \quad (2.77)$$

2.8. Weak coupling limit at fixed ν . All our considerations are founded on the expectation that various quantities related to the Bose gas simplify in **thermodynamic limit**. That is, $L \rightarrow \infty$ with $n(L) = \frac{N(L)}{L^d} \rightarrow n$. Note that we do not have a limiting object at $L = \infty$: all our statements go through the limit of a sequence of systems with finite L

On top of thermodynamic limit in this paper we consider the **weak coupling limit at fixed ν** . This means replacing v with κv and computing the lowest terms in perturbation theory in terms of κ , keeping $\nu = \kappa \hat{v}(\mathbf{0})n$ fixed.

Note that when the profile of the potential v and the parameter ν are kept fixed, then $\omega_{\text{bg}}(\mathbf{k}) = \sqrt{\frac{|\mathbf{k}|^4}{4} + \frac{\nu \hat{v}(\mathbf{k})}{\hat{v}(\mathbf{0})} |\mathbf{k}|^2}$ does not change with varying κ , and has an interesting shape, with phonons, and often also with maxons, rotons, etc. Therefore, in the limit we consider it makes then sense to assume that the Bogoliubov Hamiltonian H_{bg}^L yields a good approximation of our system at low energies. In particular, we will treat Ω_{bg} as close to the ground state of our system.

It is possible to compute the mean value of the extended zero-mode number operator on Ω_{bg} (for $d = 2, 3$)

$$n_{0,\text{bg}}(L) := \frac{1}{L^d} (\Omega_{\text{bg}} | N_{\mathbf{0}}^{\text{ext}} \Omega_{\text{bg}}) = n(L) - n_{\text{bg}}^>(L), \quad (2.78)$$

$$n_{\text{bg}}^>(L) := \frac{1}{L^d} (\Omega_{\text{bg}} | N^> \Omega_{\text{bg}}) = \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} s_{\mathbf{p}}^2. \quad (2.79)$$

We will also need the following quantity

$$\begin{aligned} & (\Omega_{\text{bg}} | e^{\mu N^>} \Omega_{\text{bg}}) \\ = & \prod_{\mathbf{p} \in \Xi_L^{>,+}} \frac{1}{\cosh^2 \theta_{\mathbf{p}}} \sum_{n,m \geq 0} \frac{(-e^\mu \tanh \theta_{\mathbf{p}})^{n+m}}{n!m!} ((a_{\mathbf{p}}^* a_{-\mathbf{p}}^*)^n | N) \otimes \Omega^> | (a_{\mathbf{p}}^* a_{-\mathbf{p}}^*)^m | N) \otimes \Omega^> \\ = & \prod_{\mathbf{p} \in \Xi_L^{>,+}} \frac{1 - \tanh^2 \theta_{\mathbf{p}}}{1 - e^{2\mu} \tanh^2 \theta_{\mathbf{p}}}, \end{aligned} \quad (2.80)$$

which is well defined only for

$$e^\mu < \frac{1}{\sup_{\mathbf{p} \in \Xi_L^{>,+}} |\tanh \theta_{\mathbf{p}}|}. \quad (2.81)$$

We remark that $n_{0,\text{bg}}(L)$ could be possibly vanishing or negative. Indeed, the operator $N_{\mathbf{0}}^{\text{ext}}$ is not the physical zero-mode number operator, and in particular its spectrum is $\mathbb{Z}$. We can compute its thermodynamic limit

$$n_{0,\text{bg}} := \lim_{L \rightarrow +\infty} n_{0,\text{bg}}(L) = n - n_{\text{bg}}^>, \quad (2.82)$$

$$n_{\text{bg}}^> := \int \frac{d\mathbf{p}}{(2\pi)^d} s_{\mathbf{p}}^2. \quad (2.83)$$

We will refer interchangeably to $n_{0,\text{bg}}(L)$ and $n_{0,\text{bg}}$ as the *extended Bogoliubov condensate density*, omitting the qualifier “finite-volume” when referring to $n_{0,\text{bg}}(L)$.

We now state another important assumption for the subsequent computation. We shall refer to it as the *condensate assumption*, since, as will become clear below, it implies that the tail of the extended Bogoliubov state Ω_{bg} is exponentially suppressed in L^{d-1} , equivalently in the number of particles.

Assumption 2.3. (Condensate assumption) Define $n_{0,\text{bg}}$ as above. We assume that

$$n_{0,\text{bg}} > 0.$$

We have $n_{0,\text{bg}} = \frac{\nu}{\kappa \hat{v}(\mathbf{0})} - n_{\text{bg}}^>$, where $n_{\text{bg}}^>$ depends only on v and ν . Therefore, Assumption 2.3 essentially says that κ is sufficiently small.

2.9. Limit $\nu \rightarrow 0$. Note that keeping ν fixed and $\kappa \rightarrow 0$ means that the density $n \rightarrow +\infty$. So this limit involves the high density regime. One can also consider the limit $\kappa \rightarrow 0$ at the constant density n . This implies $\nu \rightarrow 0$. It is then natural to rescale the momenta, obtaining a universal dispersion relation, which unfortunately does not have maxons or rotons any more:

$$\lim_{\nu \rightarrow 0} \nu^{-1} \omega_{\text{bg}}(\sqrt{\nu} \mathbf{p}, \nu) = \sqrt{\frac{1}{4} |\mathbf{p}|^4 + |\mathbf{p}|^2}, \quad (2.84)$$

$$\lim_{\nu \rightarrow 0} s_{\sqrt{\nu} \mathbf{p}} = \frac{1}{\sqrt{2}} \left(\left(1 - \left(\frac{1}{\frac{1}{2} |\mathbf{p}|^2 + 1} \right)^2 \right)^{-1/2} - 1 \right)^{1/2}, \quad (2.85)$$

$$\lim_{\nu \rightarrow 0} \nu^{-\frac{d}{2}} n_{\text{bg}}^> = \int \frac{d\mathbf{p}}{(2\pi)^d} \frac{\frac{|\mathbf{p}|^2}{2} + 1 - \sqrt{\frac{|\mathbf{p}|^4}{4} + |\mathbf{p}|^2}}{2\sqrt{\frac{|\mathbf{p}|^4}{4} + |\mathbf{p}|^2}}. \quad (2.86)$$

Thus, also in this limit $n_{\text{bg}}^>$ converges to zero, so that the condensate assumption holds. For $d = 2, 3$, (2.86) remains finite and can be computed explicitly. For $d = 3$ it equals $\frac{1}{3\pi^2}$.

The regime of small ν is closely related to the dilute limit, which will be discussed in Subsection 2.10.

2.10. Dilute limit. The material of this subsection will not be used in this paper, except for Subsect. 3.5.

As we mentioned above, the limit considered in this paper involves the high density regime. It is also interesting, and often physically more relevant, to consider dilute Bose gas. One can introduce the *dilute limit* as follows: we keep the potential v fixed (thus $\kappa = 1$), we take thermodynamic limit, and then we consider $n \rightarrow 0$. It is expected that in this limit in dimension $d = 3$ the natural parameter is not $\nu = \hat{v}(\mathbf{0})n$, it is $\mu = n4\pi\mathfrak{a}$, where $\mathfrak{a}$ is the scattering length of the potential v . Moreover, in the Bogoliubov approximation one should replace the formulas for the

dispersion relation and related quantities by

$$\omega_{\text{bg}}(\mathbf{p}, \mu) = \sqrt{\frac{1}{4}|\mathbf{p}|^4 + \mu|\mathbf{p}|^2}, \quad (2.87)$$

$$s_{\mathbf{p}} = \frac{1}{\sqrt{2}} \left(\left(1 - \left(\frac{\mu}{\frac{1}{2}|\mathbf{p}|^2 + \mu} \right)^2 \right)^{-1/2} - 1 \right)^{1/2}, \quad (2.88)$$

$$n_{\text{bg}}^> = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{\frac{|\mathbf{p}|^2}{2} + \mu - \sqrt{\frac{|\mathbf{p}|^4}{4} + |\mathbf{p}|^2\mu}}{2\sqrt{\frac{|\mathbf{p}|^4}{4} + |\mathbf{p}|^2\mu}} = \frac{\mu^{\frac{3}{2}}}{3\pi^2}. \quad (2.89)$$

Note that these formulas have essentially the same form as in the limit $\nu \rightarrow 0$ described in (2.84)–(2.86).

The two approaches (the weak coupling limit with ν fixed and the dilute limit) are consistent with one another. Indeed, in the first order of the Born approximation the scattering length satisfies $4\pi\mathbf{a} = \kappa\hat{v}(\mathbf{0}) + O(\kappa^2)$. Thus $\mu = \nu(1 + O(\kappa))$. Note also that

$$n_{0,\text{bg}} = n - \frac{1}{3\pi^2}\mu^{\frac{3}{2}} = n \left(1 - \frac{1}{3\pi^2}\sqrt{n(4\pi\mathbf{a})^3} \right), \quad (2.90)$$

so that the condition $n_{0,\text{bg}} > 0$ corresponds to

$$\sqrt{n\mathbf{a}^3} < \frac{3\sqrt{\pi}}{8}. \quad (2.91)$$

Thus for sufficiently small scattering length $\mathbf{a}$ the condensate assumption is satisfied.

In the literature on dimension 3, often, instead of the dilute limit, a seemingly different limit is considered, which is usually called the *Gross-Pitaevskii limit*. The size of the box L is kept fixed, the number of particles N goes to infinity, and the potential is rescaled as $v_N(x) := N^2v(Nx)$. Note that the scattering length depends on N as $\mathbf{a}^{(N)} := \frac{\mathbf{a}}{N}$. Thus $\mu = \frac{N}{L^3}4\pi\mathbf{a}^{(N)}$ does not change.

The Gross-Pitaevskii limit has also been referred to as the *ultra-dilute limit* [6]. Indeed, following the discussion after Theorem 1.1 in [6], one may rescale the side length of the box by a factor of N , namely $L \rightarrow NL$. The system is then equivalent to an N -particle problem with a fixed interaction potential $v(x)$ in a box of volume $O(N^3)$. Under this scaling, the vanishing-density limit $n \rightarrow 0$ is taken simultaneously with the infinite-volume and large-particle-number limits. In particular,

$$n = O(N^{-2}) \quad \text{as } N \rightarrow +\infty.$$

This differs from both our approach and the dilute limit: in both approaches the limits $\nu \rightarrow 0$, resp. $n \rightarrow 0$ are taken only after the thermodynamic limit at fixed density; see 2.13. The two procedures need not to be equivalent. Indeed, the excitation spectrum obtained in [6] retains a discrete structure, with a fixed step, whereas the perturbative expansion developed in the present paper suggests that the spectrum becomes purely continuous in the thermodynamic limit. Similarly, the ground-state energy obtained in the Gross-Pitaevskii regime in [6, Equation 1.10] differs from the prediction of the Lee-Huang-Yang formula [6, Equation 1.15].

2.11. Shift of dispersion relation. Let us now describe the main quantity computed in our paper. We will choose a sequence $\mathbf{k}_L \in \Xi_L^>$ converging to $\mathbf{k} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ and consider the perturbation theory around the sequence of vectors $b_{\mathbf{k}_L}^* \Omega_{\text{bg}}$. With a slight abuse of notation, we will drop L from $\mathbf{k}_L$.

The difference between the physical and Bogoliubov Hamiltonians, that is $H^L - H_{\text{bg}}^L$ is quite complicated and consists of several terms with different roles and different orders in κ , see (2.54).

Let us first consider the operators $R_{4,1}$ and $R_{4,2}$, which will be called the *tail terms*. They depend on the condensate degree of freedom in complicated way. In particular, they cannot be represented as good second quantized operators of the out-of-condensate degrees of freedom. In Theorem 2.4 we will show that these terms are irrelevant in thermodynamic limit for all low-lying eigenstates.

All the remaining terms in (2.54) $H_{\text{bg}}^L, H_2^L, H_3^L, H_{4,1}^L, H_{4,2}^L, H_{4,3}^L, H_{3,x}^L$ do not depend on the condensate degrees of freedom. With respect to the factorization (2.23), they can be written as

$$\mathbb{1} \otimes \tilde{H}_j^L \quad (2.92)$$

where $\tilde{H}_j^L$ is the corresponding “reduced” operator, acting on the modes in $\Xi_L^>$. In what follows, we will drop these tildes, and we will denote the reduced and unreduced operators the same way. We will treat $\Gamma_s(l^2(\Xi_L^>))$ as our basic Hilbert space.

All of these operators but $H_{3,x}^L$ do not depend on κ . The leading behavior of $H_{3,x}^L$ in terms of κ is of order 0, as we show later on. In a perturbative setting it can be expanded in a power series in κ . Thus we expect that to study Bose gas in the weak coupling limit at fixed ν we can use the perturbation theory for

$$H_{\text{bg}}^L + \kappa H_2^L + \sqrt{\kappa} H_3^L + \kappa H_{4,1}^L + \kappa H_{4,2}^L + \kappa H_{4,3}^L + \kappa \sqrt{\kappa} H_{3,x}^L \quad (2.93)$$

acting on the space $\Gamma_s(l^2(\Xi_L^>))$.

The term $\kappa \sqrt{\kappa} H_{3,x}^L$ yields contributions of a higher order in κ than the order considered here, therefore we will ignore it as well.

Thus we can use perturbation theory for

$$H_{\text{bg}}^L + \sqrt{\kappa} H_3^L + \kappa H_4^L. \quad (2.94)$$

where

$$H_4^L := H_2^L + H_{4,1}^L + H_{4,2}^L + H_{4,3}^L. \quad (2.95)$$

The shift of the dispersion relation at order κ^1 will be computed from the formula

$$\delta(\mathbf{k}) := \lim_{\epsilon \searrow 0} \lim_{L \rightarrow \infty} \left\{ (b_{\mathbf{k}}^* \Omega_{\text{bg}} | H_4^L b_{\mathbf{k}}^* \Omega_{\text{bg}}) \right. \quad (2.96)$$

$$\left. - \left(H_3^L b_{\mathbf{k}}^* \Omega_{\text{bg}} | (H_{\text{bg}}^L - \omega_{\text{bg}}(\mathbf{k}) - i\epsilon)^{-1} H_3^L b_{\mathbf{k}}^* \Omega_{\text{bg}} \right) \right. \quad (2.97)$$

$$\left. - (\Omega_{\text{bg}} | H_4^L \Omega_{\text{bg}}) + (H_3^L \Omega_{\text{bg}} | (H_{\text{bg}}^L - i\epsilon)^{-1} H_3^L \Omega_{\text{bg}}) \right\} \quad (2.98)$$

Note that it is the difference of the shift of the energy level of $b_{\mathbf{k}}^* \Omega_{\text{bg}}$ (2.96)+(2.97) and the shift for the ground state energy (2.98).

The formula for $\delta(\mathbf{k})$ can be transformed as follows:

$$\delta(\mathbf{k}) = \delta_4(\mathbf{k}) + \delta_3(\mathbf{k}), \quad (2.99)$$

$$\delta_4(\mathbf{k}) := \lim_{L \rightarrow \infty} (\Omega_{\text{bg}} | [b_{\mathbf{k}}, [H_4^L, b_{\mathbf{k}}^*]] \Omega_{\text{bg}}) \quad (2.100)$$

$$\begin{aligned} \delta_3(\mathbf{k}) := & \lim_{\epsilon \searrow 0} \lim_{L \rightarrow \infty} \left\{ - (\Omega_{\text{bg}} | [b_{\mathbf{k}}, H_3^L] (H_{\text{bg}}^L - \omega_{\text{bg}}(\mathbf{k}) - i\epsilon)^{-1} [H_3^L, b_{\mathbf{k}}^*] \Omega_{\text{bg}}) \right. \\ & - (\Omega_{\text{bg}} | [b_{\mathbf{k}}^*, H_3^L] (H_{\text{bg}}^L + \omega_{\text{bg}}(\mathbf{k}) - i\epsilon)^{-1} [H_3^L, b_{\mathbf{k}}] \Omega_{\text{bg}}) \\ & - (\Omega_{\text{bg}} | H_3^L (H_{\text{bg}}^L - i\epsilon)^{-1} [b_{\mathbf{k}}, [H_3^L, b_{\mathbf{k}}^*]] \Omega_{\text{bg}}) \\ & \left. - (\Omega_{\text{bg}} | [b_{\mathbf{k}}^*, [H_3^L, b_{\mathbf{k}}]] (H_{\text{bg}}^L - i\epsilon)^{-1} H_3^L \Omega_{\text{bg}}) \right\}. \end{aligned}$$

To obtain this we used

$$[b_{\mathbf{k}}, b_{\mathbf{k}}^*] = 1, \quad (2.101)$$

$$b_{\mathbf{k}}^*(H_{\text{bg}}^L + \omega_{\text{bg}}(\mathbf{k}) - i\epsilon)^{-1} = (H_{\text{bg}}^L - i\epsilon)^{-1} b_{\mathbf{k}}^*, \quad (2.102)$$

$$b_{\mathbf{k}}(H_{\text{bg}}^L - \omega_{\text{bg}}(\mathbf{k}) - i\epsilon)^{-1} = (H_{\text{bg}}^L - i\epsilon)^{-1} b_{\mathbf{k}}. \quad (2.103)$$

We will argue that the Hellmann-Feynman terms involving κH_2^L , $\kappa H_{4,1}^L$, $\kappa H_{4,3}^L$ vanish in thermodynamic limit. So eventually, we will use

$$H_{\text{eff}}^L := H_{\text{bg}}^L + \sqrt{\kappa} H_3^L + \kappa H_{4,2}^L. \quad (2.104)$$

on the space $\Gamma_s(l^2(\Xi_L^>))$ as an effective Hamiltonian up to the order κ^1 .

2.12. Irrelevant terms for thermodynamic limit. In this subsection we discuss various terms which vanish in thermodynamic limit, and hence do not contribute to the shift of the dispersion relation.

The tail terms $R_{4,1}$, $R_{4,2}$ formally do not depend on κ and are multiplied by κ . So, formally, they should contribute to the Hellmann-Feynman correction. However, they vanish very fast in thermodynamic limit. This is described in the theorem below, which will be shown in Sect. 5, where we will describe the mechanism which causes unphysical degrees of freedom to be invisible for large volumes.

Theorem 2.4. *Fix $d = 2, 3$ and assume the validity of Assumptions 2.1 and 2.3. Consider the normalized k -quasiparticle vector*

$$\psi_k = \mathcal{N} b_{\mathbf{k}_1}^* \cdots b_{\mathbf{k}_k}^* \Omega_{\text{bg}}, \quad \mathcal{N} = \|b_{\mathbf{k}_1}^* \cdots b_{\mathbf{k}_k}^* \Omega_{\text{bg}}\|^{-1}, \quad (2.105)$$

with momenta $\mathbf{k}_1, \dots, \mathbf{k}_k \in \Xi_L^>$ satisfying $\liminf_{L \rightarrow +\infty} |\mathbf{k}_i| > 0$. The Hellmann-Feynman corrections at order κ^1 arising from the remainder terms $R_{4,1}^L$, $R_{4,2}^L$ are zero in the thermodynamic limit. That is

$$\lim_{L \rightarrow +\infty} ((\psi_k | R_{4,1}^L \psi_k) - (\Omega_{\text{bg}} | R_{4,1}^L \Omega_{\text{bg}})) = 0, \quad (2.106)$$

$$\lim_{L \rightarrow +\infty} ((\psi_k | R_{4,2}^L \psi_k) - (\Omega_{\text{bg}} | R_{4,2}^L \Omega_{\text{bg}})) = 0. \quad (2.107)$$

The following three lemmas show that the Hellmann-Feynman contribution arising from H_2^L and $H_{4,1}^L$ and $H_{4,3}^L$ are vanishing in the thermodynamic limit. We will always assume the potential to satisfy Assumption 2.1 and $d \geq 2$.

Recall that

$$H_2^L := -\frac{1}{4L^d} \sum_{\mathbf{p} \in \Xi_L^>} \hat{v}(\mathbf{p}) A_{\mathbf{p}}^* + \text{h.c.} \quad (2.108)$$

Lemma 2.5. *We have*

$$(b_{\mathbf{k}}^* \Omega_{\text{bg}} | H_2^L b_{\mathbf{k}}^* \Omega_{\text{bg}}) - (\Omega_{\text{bg}} | H_2^L \Omega_{\text{bg}}) = O(L^{-d}). \quad (2.109)$$

Proof. We rewrite the left hand side in the commutator form and use (2.74):

$$(\Omega_{\text{bg}} | [b_{\mathbf{k}}, [H_2^L, b_{\mathbf{k}}^*]] \Omega_{\text{bg}}) = -\frac{\hat{v}(\mathbf{k})}{4L^d} 4s_{\mathbf{k}} c_{\mathbf{k}} = O(L^{-d}). \quad (2.110)$$

□

Recall that

$$H_{4,1}^L := -\frac{\hat{v}(\mathbf{0})}{2L^d} \sum_{\mathbf{p} \in \Xi_L^>} N_{\mathbf{p}}^2. \quad (2.111)$$

Lemma 2.6. *We have*

$$(b_{\mathbf{k}}^* \Omega_{\text{bg}} | H_{4,1}^L b_{\mathbf{k}}^* \Omega_{\text{bg}}) - (\Omega_{\text{bg}} | H_{4,1}^L \Omega_{\text{bg}}) = O(L^{-d}). \quad (2.112)$$

Proof. We proceed similarly as in the proof of the previous lemma, using (2.71) and (2.73):

$$(\Omega_{\text{bg}}|[b_{\mathbf{k}}, [H_{4,1}^L, b_{\mathbf{k}}^*]]\Omega_{\text{bg}}) = -\frac{\hat{v}(\mathbf{0})}{2L^d}(\Omega_{\text{bg}}|[b_{\mathbf{k}}, [N_{\mathbf{k}}^2 + N_{-\mathbf{k}}^2, b_{\mathbf{k}}^*]]\Omega_{\text{bg}}) = O(L^{-d}). \quad (2.113)$$

□

Recall that

$$H_{4,3}^L := \frac{1}{2L^d} \sum_{\substack{\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{p} \pm \mathbf{q}, \\ \mathbf{p} + \mathbf{r}, \mathbf{q} - \mathbf{r}, \mathbf{p} + \mathbf{r} - \mathbf{q} \in \Xi_L^{\geq}}} \hat{v}(\mathbf{r}) a_{\mathbf{p}+\mathbf{r}}^* a_{\mathbf{q}-\mathbf{r}}^* a_{\mathbf{q}} a_{\mathbf{p}}, \quad (2.114)$$

Lemma 2.7. *We have*

$$(b_{\mathbf{k}}^* \Omega_{\text{bg}} | H_{4,3}^L b_{\mathbf{k}}^* \Omega_{\text{bg}}) - (\Omega_{\text{bg}} | H_{4,3}^L \Omega_{\text{bg}}) = 0. \quad (2.115)$$

Proof. We can again rewrite the lhs in the commutator form:

$$(\Omega_{\text{bg}} |[b_{\mathbf{k}}, [H_{4,3}^L, b_{\mathbf{k}}^*]]\Omega_{\text{bg}}). \quad (2.116)$$

Using $b_{\mathbf{k}}^* = c_{\mathbf{k}} a_{\mathbf{k}}^* + s_{\mathbf{k}} a_{-\mathbf{k}}$ we obtain

$$[H_{4,3}^L, b_{\mathbf{k}}^*] = \frac{2c_{\mathbf{k}}}{2L^d} \sum_{\substack{\mathbf{p}, \mathbf{r}, \mathbf{p} + \mathbf{k}, \\ \mathbf{p} + \mathbf{r}, \mathbf{k} - \mathbf{r}, \mathbf{p} + \mathbf{r} - \mathbf{k} \in \Xi_L^{\geq}}} \hat{v}(\mathbf{r}) a_{\mathbf{p}+\mathbf{r}}^* a_{\mathbf{k}-\mathbf{r}}^* a_{\mathbf{p}} \quad (2.117)$$

$$+ -\frac{2s_{\mathbf{k}}}{2L^d} \sum_{\substack{\mathbf{p}, \mathbf{r}, \mathbf{p} - \mathbf{k}, \\ \mathbf{p} + \mathbf{r}, \mathbf{k} + \mathbf{r}, \mathbf{p} + \mathbf{r} + \mathbf{k} \in \Xi_L^{\geq}}} \hat{v}(\mathbf{r}) a_{\mathbf{p}}^* a_{\mathbf{p}+\mathbf{r}} a_{-\mathbf{k}-\mathbf{r}}. \quad (2.118)$$

Now the commutator of this with $b_{\mathbf{k}} = c_{\mathbf{k}} a_{\mathbf{k}} + s_{\mathbf{k}} a_{-\mathbf{k}}^*$ is zero. □

Remark 2.8. (Naive thermodynamic limit) Let us try to describe the thermodynamic limit naively. Remember that we consider $L \rightarrow \infty$ and we keep ν and $\mathbf{k} \mapsto \hat{v}(\mathbf{k})$ fixed. Formally, $\Xi_L^{\geq}$ converges to $\mathbb{R}^d$. $\frac{1}{L^d} \sum_{\mathbf{k} \in \Xi_L^{\geq}}$ should be replaced with

$\frac{1}{(2\pi)^d} \int d\mathbf{k}$. The operators $a_{\mathbf{k}}$, $a_{\mathbf{k}}^*$ for discrete $\mathbf{k}$ should be replaced by $\frac{(2\pi)^{\frac{d}{2}}}{L^{\frac{d}{2}}} a_{\mathbf{k}}$, $\frac{(2\pi)^{\frac{d}{2}}}{L^{\frac{d}{2}}} a_{\mathbf{k}}^*$ for continuous $\mathbf{k}$. Thus H_{bg}^L formally converges to

$$H_{\text{bg}} := \int d\mathbf{p} \left(\frac{\mathbf{p}^2}{2} + \frac{\nu \hat{v}(\mathbf{p})}{\hat{v}(\mathbf{0})} \right) a_{\mathbf{p}}^* a_{\mathbf{p}} + \int d\mathbf{p} \left(\frac{\nu \hat{v}(\mathbf{p})}{2\hat{v}(\mathbf{0})} a_{\mathbf{p}} a_{-\mathbf{p}} + \text{h. c.} \right). \quad (2.119)$$

Similarly, in thermodynamic limit H_3^L and $H_{4,3}^L$ formally converge to

$$H_3 := \frac{1}{(2\pi)^{\frac{d}{2}}} \int d\mathbf{k} d\mathbf{p} \frac{\sqrt{\nu} \hat{v}(\mathbf{k})}{\sqrt{\hat{v}(\mathbf{0})}} (a_{\mathbf{p}+\mathbf{k}}^* a_{\mathbf{k}} a_{\mathbf{p}} + \text{h. c.}), \quad (2.120)$$

$$H_{4,3} := \frac{1}{2(2\pi)^d} \int d\mathbf{p} d\mathbf{k} d\mathbf{q} \hat{v}(\mathbf{k}) a_{\mathbf{p}+\mathbf{k}}^* a_{\mathbf{q}-\mathbf{k}}^* a_{\mathbf{q}} a_{\mathbf{p}}.$$

However, as we have just noted, $H_{4,3}^L$ is of little relevance for our analysis. Formally, in thermodynamic limit $H_{4,2}^L$ and $H_{4,1}^L$ converge to zero. However, $H_{4,2}^L$ plays an important role in calculations of the quasiparticle spectrum.

This illustrates the fact that thermodynamic limit in our problem is quite subtle.

Finally, let us now consider $H_{3,x}^L$. This Hamiltonian contains a factor

$$\left| 1 - \frac{N^>}{N} \right|_+^{\frac{1}{2}} - 1 = \left| 1 - \frac{\kappa \hat{v}(\mathbf{0})}{\nu L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} N_{\mathbf{p}} \right|_+^{\frac{1}{2}} - 1.$$

As we will now show, the matrix elements of this operator in low energy states contribute at least as an order κ^1

Lemma 2.9. *Suppose that the condensate assumption 2.3 is satisfied. Take momenta $\mathbf{q}_1, \dots, \mathbf{q}_m; \mathbf{k}_1, \dots, \mathbf{k}_n \in \Xi_L^\geq$. Then, we have*

$$\left| \left(b_{\mathbf{q}_1}^* \dots b_{\mathbf{q}_m}^* \Omega_{\text{bg}} \left(\left| 1 - \frac{N^>}{N} \right|_+^{\frac{1}{2}} - 1 \right) b_{\mathbf{k}_1}^* \dots b_{\mathbf{k}_n}^* \Omega_{\text{bg}} \right) \right| = O(\kappa). \quad (2.121)$$

Proof. We have the operator inequality

$$\left| 1 - \left| 1 - \frac{N^>}{N} \right|_+^{\frac{1}{2}} \right| \leq \frac{N^>}{N} = \kappa \frac{\hat{v}(\mathbf{0})}{\nu L^d} \sum_{\mathbf{p} \in \Xi_L^\geq} N_{\mathbf{p}} \quad (2.122)$$

If we restrict the analysis to states with a finite number of Bogoliubov excitations, the excited number operator $\sum_{\mathbf{p} \in \Xi_L^\geq} N_{\mathbf{p}}$ is bounded on the corresponding finite-particle subspace and thus the matrix elements are well defined and of order $O(\kappa)$ as $\kappa \rightarrow 0$. $\square$

3. COMPUTING PERTURBATIVE CORRECTIONS

In this section we compute perturbative corrections of order κ^1 to the dispersion relation. They consist of two terms: the Hellmann-Feynman correction and the real part of the Fermi golden rule correction. We also extract the leading behavior of these corrections in the low momentum limit, which yield the phonon velocity. In this section we content ourselves with formal calculations—rigorous estimates are deferred to Section 6.

In addition to Assumptions 2.1 and 2.3, we impose the following regularity conditions, which are needed for the detailed computation of the real energy shift.

Assumption 3.1. Suppose there exists a $\nu_0 \in]0, +\infty[$ such that the potential satisfies the following assumptions for all the possible choices of $\nu \in]0, \nu_0]$

$$v \text{ is rotationally invariant;} \quad (3.1a)$$

$$\hat{v} \text{ is } C^5 \text{ in a neighborhood of } \mathbf{k} = \mathbf{0}. \quad (3.1b)$$

$$\hat{v} \text{ is } C^2 \text{ and } \partial_i \partial_j \hat{v} \text{ is bounded for every } i, j \in \{1, 2, \dots, d\}; \quad (3.1c)$$

$$\partial_i \hat{v} \text{ and } \partial_i \partial_j \hat{v} \in L^2(\mathbb{R}^d), \text{ for every } i, j \in \{1, 2, \dots, d\}. \quad (3.1d)$$

$$\frac{\nu}{\hat{v}(\mathbf{0})} \frac{d^2 \hat{v}}{dk^2}(\mathbf{0}) > -\frac{1}{2}. \quad (3.1e)$$

We remark that under these hypotheses

$$\hat{v}(\mathbf{k}) - \hat{v}(\mathbf{0}) = O(|\mathbf{k}|^2), \text{ as } \mathbf{k} \rightarrow \mathbf{0}. \quad (3.2)$$

Since the functions considered below are rotationally invariant, we sometimes identify a function $f(\mathbf{p})$ with the corresponding radial function $f(p)$, where $p = |\mathbf{p}|$; for example, we write $\omega_{\text{bg}}(\mathbf{p}) = \omega_{\text{bg}}(p)$ and $\hat{v}(\mathbf{p}) = \hat{v}(p)$. When no confusion can arise, primes denote derivatives with respect to the scalar argument:

$$\frac{d}{dx} f(x) \equiv f'(x). \quad (3.3)$$

3.1. Hellmann-Feynman correction. This section is devoted to the Hellmann-Feynman term $\delta_4^{\text{can}}(\mathbf{k}, \nu, v)$ defined in (2.100). As we argued, in this term we can replace H_4^L with $H_{4,2}^L$. Thus we need to compute

$$\delta_4^{\text{can}}(\mathbf{k}, \nu, \hat{v}) := \lim_{L \rightarrow \infty} \left((b_{\mathbf{k}}^* \Omega_{\text{bg}} | H_{4,2}^L b_{\mathbf{k}}^* \Omega_{\text{bg}}) - (\Omega_{\text{bg}} | H_{4,2}^L \Omega_{\text{bg}}) \right) \quad (3.4)$$

$$= \lim_{L \rightarrow \infty} (\Omega_{\text{bg}} | [b_{\mathbf{k}}, [H_{4,2}^L, b_{\mathbf{k}}^*]] \Omega_{\text{bg}}). \quad (3.5)$$

Recall that

$$H_{4,2}^L := H_{NN}^L + H_{NA}^L + H_{AA}^L, \quad (3.6)$$

$$H_{NN}^L := -\frac{1}{2L^d} \sum_{\mathbf{q} \in \Xi_L^{\geq}} N_{\mathbf{q}} \sum_{\mathbf{p} \in \Xi_L^{\geq}} N_{\mathbf{p}} (\hat{v}(\mathbf{p}) + \hat{v}(\mathbf{q}) - \hat{v}(\mathbf{p} - \mathbf{q})),$$

$$H_{NA}^L := -\frac{1}{2L^d} \sum_{\mathbf{q} \in \Xi_L^{\geq}} N_{\mathbf{q}} \sum_{\mathbf{p} \in \Xi_L^{\geq}} \hat{v}(\mathbf{p}) A_{\mathbf{p}}^* + \text{h.c.},$$

$$H_{AA}^L := \frac{1}{2L^d} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{p} \pm \mathbf{q} \in \Xi_L^{\geq}} \hat{v}(\mathbf{p} - \mathbf{q}) A_{\mathbf{p}}^* A_{\mathbf{q}}.$$

To evaluate the NN term in $H_{4,2}^L$ we compute

$$[b_{\mathbf{k}}, [N_{\mathbf{q}} N_{\mathbf{p}}, b_{\mathbf{k}}^*]] = N_{\mathbf{q}} [b_{\mathbf{k}}, [N_{\mathbf{p}}, b_{\mathbf{k}}^*]] + [b_{\mathbf{k}}, [N_{\mathbf{q}}, b_{\mathbf{k}}^*]] N_{\mathbf{p}} \quad (3.7)$$

$$+ [b_{\mathbf{k}}, N_{\mathbf{q}}] [N_{\mathbf{p}}, b_{\mathbf{k}}^*] + [b_{\mathbf{k}}^*, N_{\mathbf{q}}] [N_{\mathbf{p}}, b_{\mathbf{k}}]. \quad (3.8)$$

The terms in (3.8) contribute $O(L^{-d})$, so that it will be dropped. Thus we obtain

$$\begin{aligned} & \left(\Omega_{\text{bg}} | \left[b_{\mathbf{k}}, \left[H_{NN}^L, b_{\mathbf{k}}^* \right] \right] \Omega_{\text{bg}} \right) \\ &= -\frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} s_{\mathbf{p}}^2 c_{\mathbf{k}}^2 (\hat{v}(\mathbf{p}) + \hat{v}(\mathbf{k}) - \hat{v}(\mathbf{p} - \mathbf{k})) \\ & \quad - \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} s_{\mathbf{p}}^2 s_{\mathbf{k}}^2 (\hat{v}(\mathbf{p}) + \hat{v}(\mathbf{k}) - \hat{v}(\mathbf{p} + \mathbf{k})) + O(L^{-d}). \end{aligned} \quad (3.9)$$

Similarly, the NA term gives

$$\begin{aligned} & (\Omega_{\text{bg}} | [b_{\mathbf{k}}, [H_{NA}^L, b_{\mathbf{k}}^*]] \Omega_{\text{bg}}) \\ &= \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} 2s_{\mathbf{p}}^2 c_{\mathbf{k}} s_{\mathbf{k}} \hat{v}(\mathbf{k}) + \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} (c_{\mathbf{k}}^2 + s_{\mathbf{k}}^2) c_{\mathbf{p}} s_{\mathbf{p}} \hat{v}(\mathbf{p}) + O(L^{-d}). \end{aligned} \quad (3.10)$$

Finally, the AA term gives

$$\begin{aligned} & (\Omega_{\text{bg}} | [b_{\mathbf{k}}, [H_{AA}^L, b_{\mathbf{k}}^*]] \Omega_{\text{bg}}) \\ &= \frac{1}{L^d} \sum_{\mathbf{p}, \mathbf{p} \pm \mathbf{k} \in \Xi_L^{\geq}} c_{\mathbf{k}} s_{\mathbf{k}} c_{\mathbf{p}} s_{\mathbf{p}} (\hat{v}(\mathbf{p} + \mathbf{k}) + \hat{v}(\mathbf{p} - \mathbf{k})) + O(L^{-d}) \end{aligned} \quad (3.11)$$

Combining these contributions and using $s_{\mathbf{k}} = s_{-\mathbf{k}}$, $c_{\mathbf{k}} = c_{-\mathbf{k}}$, we obtain

$$\begin{aligned} & (\Omega_{\text{bg}} | [b_{\mathbf{k}}, [H_{4,2}^L, b_{\mathbf{k}}^*]] \Omega_{\text{bg}}) = \\ & = (c_{\mathbf{k}} + s_{\mathbf{k}})^2 \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} c_{\mathbf{p}} s_{\mathbf{p}} \hat{v}(\mathbf{p}) \end{aligned} \quad (3.12)$$

$$+ (c_{\mathbf{k}}^2 + s_{\mathbf{k}}^2) \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} s_{\mathbf{p}}^2 \left(-\hat{v}(\mathbf{p}) + \frac{\hat{v}(\mathbf{p} + \mathbf{k}) + \hat{v}(\mathbf{p} - \mathbf{k})}{2} \right) \quad (3.13)$$

$$+ 2c_{\mathbf{k}} s_{\mathbf{k}} \frac{1}{L^d} \sum_{\mathbf{p}, \mathbf{p} \pm \mathbf{k} \in \Xi_L^{\geq}} c_{\mathbf{p}} s_{\mathbf{p}} \left(-\hat{v}(\mathbf{p}) + \frac{\hat{v}(\mathbf{p} + \mathbf{k}) + \hat{v}(\mathbf{p} - \mathbf{k})}{2} \right) \quad (3.14)$$

$$- (c_{\mathbf{k}} - s_{\mathbf{k}})^2 \hat{v}(\mathbf{k}) \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} s_{\mathbf{p}}^2 + O(L^{-d}). \quad (3.15)$$

These expressions have a thermodynamic limit:

$$\delta_4^{\text{can}}(\mathbf{k}, \nu, \hat{v}) = \lim_{L \rightarrow +\infty} (\Omega | [b_{\mathbf{k}}, [H_{4,2}^L, b_{\mathbf{k}}^*]] \Omega) \quad (3.16)$$

$$= (c_{\mathbf{k}} + s_{\mathbf{k}})^2 \frac{1}{(2\pi)^d} \int d\mathbf{p} c_{\mathbf{p}} s_{\mathbf{p}} \hat{v}(\mathbf{p}) \quad (3.17)$$

$$+ (c_{\mathbf{k}}^2 + s_{\mathbf{k}}^2) \frac{1}{(2\pi)^d} \int d\mathbf{p} s_{\mathbf{p}}^2 \left(-\hat{v}(\mathbf{p}) + \frac{\hat{v}(\mathbf{p} + \mathbf{k}) + \hat{v}(\mathbf{p} - \mathbf{k})}{2} \right) \quad (3.18)$$

$$+ 2c_{\mathbf{k}} s_{\mathbf{k}} \frac{1}{(2\pi)^d} \int d\mathbf{p} c_{\mathbf{p}} s_{\mathbf{p}} \left(-\hat{v}(\mathbf{p}) + \frac{\hat{v}(\mathbf{p} + \mathbf{k}) + \hat{v}(\mathbf{p} - \mathbf{k})}{2} \right) \quad (3.19)$$

$$- (c_{\mathbf{k}} - s_{\mathbf{k}})^2 \hat{v}(\mathbf{k}) \frac{1}{(2\pi)^d} \int d\mathbf{p} s_{\mathbf{p}}^2. \quad (3.20)$$

We can then express $c_{\mathbf{k}}$, $s_{\mathbf{k}}$ in simpler quantities using

$$c_{\mathbf{k}} s_{\mathbf{k}} = \frac{\hat{v}(\mathbf{k}) \nu}{\hat{v}(\mathbf{0}) 2\omega_{\text{bg}}(\mathbf{k})}, \quad (c_{\mathbf{k}} - s_{\mathbf{k}})^2 = \frac{\mathbf{k}^2}{2\omega_{\text{bg}}(\mathbf{k})}, \quad (c_{\mathbf{k}} + s_{\mathbf{k}})^2 = \frac{2\omega_{\text{bg}}(\mathbf{k})}{\mathbf{k}^2}, \quad (3.21)$$

$$c_{\mathbf{k}}^2 + s_{\mathbf{k}}^2 = \frac{\frac{\mathbf{k}^2}{2} + \frac{\hat{v}(\mathbf{k})}{\hat{v}(\mathbf{0})} \nu}{\omega_{\text{bg}}(\mathbf{k})}, \quad s_{\mathbf{k}}^2 = \frac{\frac{\mathbf{k}^2}{2} + \frac{\hat{v}(\mathbf{k})}{\hat{v}(\mathbf{0})} \nu - \omega_{\text{bg}}(\mathbf{k})}{2\omega_{\text{bg}}(\mathbf{k})}. \quad (3.22)$$

From this point onward, we fix the spatial dimension $d = 3$. The above computations lead to the following infrared asymptotics of the Hellman–Feynman correction, which we will prove rigorously in Section 6.

Proposition 3.2. *Suppose that the potential satisfies Assumptions 2.1 and 3.1. Then*

$$\delta_4^{\text{can}}(\mathbf{k}, \nu, \hat{v}) = c_{\text{ir}}(k, \nu, \hat{v}) + k c_4(\nu, \hat{v}) + o\left(\frac{k}{\sqrt{\nu}}\right), \quad (3.23)$$

as $\frac{k}{\sqrt{\nu}} \rightarrow 0$, where the terms c_{ir} and c_4 are equal to

$$c_{\text{ir}}(k, \nu, \hat{v}) := \frac{\omega_{\text{bg}}(k)}{k^2} \int_0^{+\infty} \frac{dp}{2\pi^2} p^2 \frac{\hat{v}(p)^2 \nu}{\hat{v}(0) \omega_{\text{bg}}(p)}, \quad (3.24)$$

$$c_4(\nu, \hat{v}) = \frac{\sqrt{\nu}}{12\pi^2} \int_0^\infty dp \frac{\left(\frac{p^2}{2} + \frac{2\hat{v}(p)\nu}{\hat{v}(0)} - \omega_{\text{bg}}(p)\right)}{2\omega_{\text{bg}}(p)} (p^2 \hat{v}''(p) + 2p \hat{v}'(p)) \quad (3.25)$$

$$- \frac{\hat{v}(0)}{4\pi^2 \sqrt{\nu}} \int_0^\infty dp \frac{\left(\frac{p^2}{2} + \frac{\hat{v}(p)}{\hat{v}(0)} \nu - \omega_{\text{bg}}(p)\right) p^2}{2\omega_{\text{bg}}(p)}. \quad (3.26)$$

Note that the term c_{ir} behaves as $O(k^{-1})$. It will be cancelled by an identical term coming from the Fermi golden rule.

3.2. Fermi golden rule correction. In this section we will analyze the real part of the Fermi golden rule correction to the dispersion relation (we take $\delta_3^{\text{can}} \equiv \text{Re}\delta_3^{\text{can}}$):

$$\delta_3^{\text{can}}(\mathbf{k}, \nu, \hat{v}) = \lim_{\epsilon \searrow 0} \lim_{L \rightarrow +\infty} \text{Re} \left[-(b_{\mathbf{k}}^* \Omega_{\text{bg}} | H_3^L (H_{\text{bg}}^L - \omega_{\text{bg}}(\mathbf{k}) - i\epsilon)^{-1} H_3^L b_{\mathbf{k}}^* \Omega_{\text{bg}}) \right. \quad (3.27)$$

$$\left. + (\Omega_{\text{bg}} | H_3^L (H_{\text{bg}}^L - i\epsilon)^{-1} H_3^L \Omega_{\text{bg}}) \right] \quad (3.28)$$

In particular, we will estimate its behavior for small $\frac{|\mathbf{k}|}{\sqrt{\nu}} \rightarrow 0$. The integral representation for δ_3^{can} is derived in [23]:

$$\delta_3^{\text{can}}(\mathbf{k}, \nu, \hat{v}) = -\frac{1}{2} \text{Re} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{j(\mathbf{k}; \mathbf{p}, \mathbf{p} - \mathbf{k})^2}{\omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p} - \mathbf{k}) - \omega_{\text{bg}}(\mathbf{k}) - i0} \quad (3.29)$$

$$- \frac{1}{2} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{\kappa(\mathbf{k}, \mathbf{p}, \mathbf{p} - \mathbf{k})^2}{\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p} - \mathbf{k})}. \quad (3.30)$$

The function $j(\mathbf{k}; \mathbf{p}, \mathbf{q})$ contained in (3.29) is defined as

$$j(\mathbf{k}; \mathbf{p}, \mathbf{q}) = \sqrt{\frac{\nu}{\hat{v}(\mathbf{0})}} (-\hat{v}(\mathbf{k})(c_{\mathbf{k}} - s_{\mathbf{k}})(c_{\mathbf{p}} s_{\mathbf{q}} + s_{\mathbf{p}} c_{\mathbf{q}}) \\ + \hat{v}(\mathbf{p})(c_{\mathbf{p}} - s_{\mathbf{p}})(c_{\mathbf{q}} c_{\mathbf{k}} + s_{\mathbf{q}} s_{\mathbf{k}}) + \hat{v}(\mathbf{q})(c_{\mathbf{q}} - s_{\mathbf{q}})(c_{\mathbf{p}} c_{\mathbf{k}} + s_{\mathbf{p}} s_{\mathbf{k}})). \quad (3.31)$$

j is symmetric under interchange of its last two arguments, i.e. $j(\mathbf{k}; \mathbf{p}, \mathbf{q}) = j(\mathbf{k}; \mathbf{q}, \mathbf{p})$. The second function, $\kappa(\mathbf{k}, \mathbf{p}, \mathbf{q})$, which appears inside the second integral in Equation (3.30), is

$$\kappa(\mathbf{k}, \mathbf{p}, \mathbf{q}) = \sqrt{\frac{\nu}{\hat{v}(\mathbf{0})}} (-\hat{v}(\mathbf{k})(c_{\mathbf{k}} - s_{\mathbf{k}})(c_{\mathbf{p}} s_{\mathbf{q}} + s_{\mathbf{p}} c_{\mathbf{q}}) \\ - \hat{v}(\mathbf{p})(c_{\mathbf{p}} - s_{\mathbf{p}})(c_{\mathbf{q}} s_{\mathbf{k}} + s_{\mathbf{q}} c_{\mathbf{k}}) - \hat{v}(\mathbf{q})(c_{\mathbf{q}} - s_{\mathbf{q}})(s_{\mathbf{p}} c_{\mathbf{k}} + c_{\mathbf{p}} s_{\mathbf{k}})). \quad (3.32)$$

κ is symmetric under permutations of its three arguments; $\kappa(\mathbf{k}, \mathbf{p}, \mathbf{q}) = \kappa(\mathbf{k}, \mathbf{q}, \mathbf{p}) = \kappa(\mathbf{q}, \mathbf{k}, \mathbf{p})$.

A rigorous analysis of the integrals (3.29) and (3.30) requires lengthy estimates of the relevant contributions and their remainders. We therefore begin with a formal derivation of their asymptotic structure as $\frac{|\mathbf{k}|}{\sqrt{\nu}} \rightarrow 0$. In particular, in the following formal derivation we forget about the principal value in δ_3^{can} , assuming that it does not cause any issue. This is justified in Proposition 6.16.

We use the notation

$$k := |\mathbf{k}|, \quad p := |\mathbf{p}|, \quad q := |\mathbf{p} - \mathbf{k}|. \quad (3.33)$$

The triangle inequality gives

$$-k \leq p - q \leq k, \quad (3.34)$$

which motivates the following change of variables.

Definition 3.3. (t - s coordinates) We define

$$t = \frac{p+q}{2}, \quad s = \frac{p-q}{2k}, \quad (3.35)$$

or, equivalently,

$$p = t + ks, \quad q = t - ks, \quad (3.36)$$

with

$$t \in \left[\frac{k}{2}, +\infty \right[, \quad s \in \left[-\frac{1}{2}, \frac{1}{2} \right]. \quad (3.37)$$

Under this change of variables, the symmetry $p \leftrightarrow q$ becomes $s \leftrightarrow -s$, while the measure transforms as

$$d\mathbf{p} = \frac{pq}{k} dp dq d\varphi = 2pq d\varphi dt ds. \quad (3.38)$$

Consequently,

$$\delta_3^{\text{can}}(\mathbf{k}, \nu, \hat{v}) = - \int_{\frac{k}{2}}^{+\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dt ds}{4\pi^2} \frac{j(k; p, q)^2 pq}{\omega_{\text{bg}}(p) + \omega_{\text{bg}}(q) - \omega_{\text{bg}}(k)} \quad (3.39)$$

$$- \int_{\frac{k}{2}}^{+\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dt ds}{4\pi^2} \frac{\kappa(k, p, q)^2 pq}{\omega_{\text{bg}}(p) + \omega_{\text{bg}}(q) + \omega_{\text{bg}}(k)}. \quad (3.40)$$

The two interaction kernels can be decomposed as

$$j(k; p, q) = \frac{1}{2} \sqrt{\frac{\nu}{\hat{v}(0)}} \left[(c_k + s_k) j_+(p, q) + (c_k - s_k) j_-(k, p, q) \right], \quad (3.41)$$

$$\kappa(k, p, q) = \frac{1}{2} \sqrt{\frac{\nu}{\hat{v}(0)}} \left[-(c_k + s_k) j_+(p, q) + (c_k - s_k) j_-(k, p, q) \right], \quad (3.42)$$

where

$$j_+(p, q) = (c_p c_q - s_p s_q) (\hat{v}(p) + \hat{v}(q)) + (c_p s_q - s_p c_q) (\hat{v}(p) - \hat{v}(q)), \quad (3.43)$$

$$j_-(k, p, q) = (c_p c_q + s_p s_q) (\hat{v}(p) + \hat{v}(q)) - (2\hat{v}(k) + \hat{v}(p) + \hat{v}(q)) (c_p s_q + s_p c_q). \quad (3.44)$$

Using $(c_k + s_k)(c_k - s_k) = 1$, we can therefore combine the two Fermi golden rule channels as

$$\begin{aligned} \delta_3^{\text{can}}(\mathbf{k}, \nu, \hat{v}) = & - \frac{\nu}{16\pi^2 \hat{v}(0)} \int_{\frac{k}{2}}^{+\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} dt ds pq \left[\right. \\ & (c_k + s_k)^2 j_+(p, q)^2 \frac{2(\omega_{\text{bg}}(p) + \omega_{\text{bg}}(q))}{(\omega_{\text{bg}}(p) + \omega_{\text{bg}}(q))^2 - \omega_{\text{bg}}(k)^2} \\ & + 2j_+(p, q) j_-(k, p, q) \frac{2\omega_{\text{bg}}(k)}{(\omega_{\text{bg}}(p) + \omega_{\text{bg}}(q))^2 - \omega_{\text{bg}}(k)^2} \\ & \left. + (c_k - s_k)^2 j_-(k, p, q)^2 \frac{2(\omega_{\text{bg}}(p) + \omega_{\text{bg}}(q))}{(\omega_{\text{bg}}(p) + \omega_{\text{bg}}(q))^2 - \omega_{\text{bg}}(k)^2} \right]. \end{aligned} \quad (3.45)$$

We have

$$(c_k + s_k)^2 = \frac{2\sqrt{\nu}}{k} + O\left(\frac{k}{\sqrt{\nu}}\right), \quad (3.46)$$

$$\omega_{\text{bg}}(k) = \sqrt{\nu} k + O\left(\frac{k^2}{\nu}\right), \quad (3.47)$$

$$(c_k - s_k)^2 = \frac{k}{2\sqrt{\nu}} + o\left(\frac{k}{\sqrt{\nu}}\right). \quad (3.48)$$

The dominant contributions to the integrals in δ_3^{can} will come from the diagonal $p = q = t$. For j_- it will suffice to use

$$j_-(k, p, q) = j_-(0, t, t) + O\left(\frac{k}{\sqrt{\nu}}\right), \quad (3.49)$$

$$j_-(0, t, t) = \frac{\hat{v}(t)(t^2 - 2\nu)}{\omega_{\text{bg}}(t)}. \quad (3.50)$$

For j_+ , because of the more singular prefactor (3.46), we will need a more precise asymptotics around the diagonal:

$$j_+(p, q) = j_+(t, t) + \frac{k^2 s^2}{2} (\partial_p - \partial_q)^2 j_+(p, q) \Big|_{p=q=t} + o\left(\frac{k^2}{\nu}\right), \quad (3.51)$$

$$j_+(t, t) = 2\hat{v}(t), \quad (3.52)$$

$$\begin{aligned} (\partial_p - \partial_q)^2 j_+(p, q) \Big|_{p=q=t} &= 2\hat{v}''(t) - 8\hat{v}'(t) \left(\frac{\omega'_{\text{bg}}(t)}{2\omega_{\text{bg}}(t)} - \frac{1}{t} \right) \\ &\quad + 8\hat{v}(t) \left(\frac{\omega'_{\text{bg}}(t)}{2\omega_{\text{bg}}(t)} - \frac{1}{t} \right)^2. \end{aligned} \quad (3.53)$$

The identity (3.53) in Lemma in Subsection 3.4 below.

The remaining factors are symmetric under $p \leftrightarrow q$. Hence, in the t - s coordinates, their formal expansions contain only even powers of k :

$$pq = t^2 - k^2 s^2, \quad (3.54)$$

$$\omega_{\text{bg}}(p) + \omega_{\text{bg}}(q) = 2\omega_{\text{bg}}(t) + k^2 s^2 \omega''_{\text{bg}}(t) + o(k^2), \quad (3.55)$$

$$\begin{aligned} \frac{2(\omega_{\text{bg}}(p) + \omega_{\text{bg}}(q))}{(\omega_{\text{bg}}(p) + \omega_{\text{bg}}(q))^2 - \omega_{\text{bg}}(k)^2} &= \frac{1}{\omega_{\text{bg}}(t)} + \frac{\nu k^2}{4\omega_{\text{bg}}(t)^3} \\ &\quad - \frac{k^2 s^2 \omega''_{\text{bg}}(t)}{2\omega_{\text{bg}}(t)^2} + o\left(\frac{k^2}{\nu}\right), \end{aligned} \quad (3.56)$$

$$\frac{1}{(\omega_{\text{bg}}(p) + \omega_{\text{bg}}(q))^2 - \omega_{\text{bg}}(k)^2} = \frac{1}{4\omega_{\text{bg}}(t)^2} + O\left(\frac{k^2}{\nu}\right). \quad (3.57)$$

Keeping the terms of the order k^{-1} and k in (3.45), we expect the following asymptotics for small k :

$$\delta_3^{\text{can}}(\mathbf{k}, \nu, \hat{v}) = -c_{\text{ir}}(k, \nu, v) + k c_3(\nu, \hat{v}) + o\left(\frac{k}{\sqrt{\nu}}\right), \quad (3.58)$$

where

$$c_{\text{ir}}(k, \nu, \hat{v}) = \frac{\nu}{16\pi^2 \hat{v}(0)} (c_k + s_k)^2 \int_0^{+\infty} dt t^2 \frac{j_+(t, t)^2}{\omega_{\text{bg}}(t)} \quad (3.59)$$

$$= \frac{\nu \omega_{\text{bg}}(k)}{2k^2 \pi^2 \hat{v}(0)} \int_0^{+\infty} dt \frac{t^2 \hat{v}(t)^2}{\omega_{\text{bg}}(t)} \quad (3.60)$$

is the term of order $O(k^{-1})$, the same as in (3.24), except that it has the opposite sign as in the Hellman–Feynman correction, and

$$c_3(\nu, \hat{v}) = \frac{\nu \hat{v}(0)}{16\pi^2} - \frac{\nu}{16\pi^2 \hat{v}(0)} \int_0^{+\infty} dt \quad (3.61)$$

$$\left[2\sqrt{\nu} j_+(t, t)^2 \left(-\frac{1}{12\omega_{\text{bg}}(t)} + \frac{\nu t^2}{4\omega_{\text{bg}}(t)^3} - \frac{t^2 \omega''_{\text{bg}}(t)}{24\omega_{\text{bg}}(t)^2} \right) \right. \quad (3.62)$$

$$\left. + \sqrt{\nu} \frac{t^2 j_+(t, t)}{6\omega_{\text{bg}}(t)} (\partial_p - \partial_q)^2 j_+(p, q) \Big|_{p=q=t} \right. \quad (3.63)$$

$$\left. + \sqrt{\nu} \frac{j_+(t, t) j_-(0, t, t) t^2}{\omega_{\text{bg}}(t)^2} + \frac{1}{\sqrt{\nu}} \frac{j_-(0, t, t)^2 t^2}{2\omega_{\text{bg}}(t)} \right] \quad (3.64)$$

is the Fermi golden rule contribution to the phonon velocity correction. In the above, we used $\int_{-\frac{1}{2}}^{\frac{1}{2}} s^2 ds = \frac{1}{12}$. The first term on the right of (3.61) (without

integration) comes from the integral of the infrared term (3.60) in the region $[0, \frac{k}{2}]$:

$$\frac{\nu\omega_{\text{bg}}(k)}{2k^2\pi^2\hat{v}(0)} \int_0^{\frac{k}{2}} dt \frac{t^2\hat{v}(t)^2}{\omega_{\text{bg}}(t)} = k \frac{\nu\hat{v}(0)}{16\pi^2} + o\left(\frac{k}{\sqrt{\nu}}\right). \quad (3.65)$$

Next we observe that

$$\begin{aligned} & \frac{\nu^{3/2}t^2j_+(t,t)^2}{2\omega_{\text{bg}}(t)^3} + \sqrt{\nu} \frac{t^2j_+(t,t)j_-(0,t,t)}{\omega_{\text{bg}}(t)^2} + \frac{t^2j_-(0,t,t)^2}{2\sqrt{\nu}\omega_{\text{bg}}(t)} \\ &= \frac{t^6\hat{v}(t)^2}{2\sqrt{\nu}\omega_{\text{bg}}(t)^3}. \end{aligned} \quad (3.66)$$

Using (3.66), inserting (3.53) we obtain

$$c_3(\nu, \hat{v}) = \frac{\nu\hat{v}(0)}{16\pi^2} - \frac{\nu}{16\pi^2\hat{v}(0)} \int_0^{+\infty} dt \quad (3.67)$$

$$\left[-\frac{\sqrt{\nu}t^2\hat{v}(t)^2\omega_{\text{bg}}''(t)}{3\omega_{\text{bg}}(t)^2} + \frac{2\sqrt{\nu}t^2\hat{v}(t)}{3\omega_{\text{bg}}(t)} \left(\hat{v}''(t) - 4\hat{v}'(t) \left(\frac{\omega_{\text{bg}}'(t)}{2\omega_{\text{bg}}(t)} - \frac{1}{t} \right) \right) \right] \quad (3.68)$$

$$+ \frac{2\sqrt{\nu}\hat{v}(t)^2}{3\omega_{\text{bg}}(t)} \left(-1 + 4t^2 \left(\frac{\omega_{\text{bg}}'(t)}{2\omega_{\text{bg}}(t)} - \frac{1}{t} \right)^2 \right) + \frac{t^6\hat{v}(t)^2}{2\nu^{\frac{1}{2}}\omega_{\text{bg}}(t)^3} \Big]. \quad (3.69)$$

The integral in the preceding expression is absolutely convergent. The second term in (3.69) (which comes from (3.66)) is regular at origin.

Let us look at the first term of (3.69). By the regularity and radial symmetry of the potential,

$$\hat{v}(t) = \hat{v}(0) + \frac{\hat{v}''(0)}{2}t^2 + O(t^4), \quad (3.70)$$

while

$$\omega_{\text{bg}}(t) = \sqrt{\nu}t + \left(\frac{1}{8\sqrt{\nu}} + \frac{\sqrt{\nu}\hat{v}''(0)}{4\hat{v}(0)} \right) t^3 + O(t^5). \quad (3.71)$$

It follows that

$$\frac{\omega_{\text{bg}}'(t)}{2\omega_{\text{bg}}(t)} - \frac{1}{t} = -\frac{1}{2t} + O(t), \quad \omega_{\text{bg}}''(t) = O(t), \quad (3.72)$$

and therefore,

$$-1 + 4t^2 \left(\frac{\omega_{\text{bg}}'(t)}{2\omega_{\text{bg}}(t)} - \frac{1}{t} \right)^2 = O(t^2). \quad (3.73)$$

Thus the integrand in (3.69) is $O(t)$ at origin.

We next consider the behavior at infinity $t \rightarrow +\infty$. In this regime,

$$\omega_{\text{bg}}(t) = O(t^2), \quad \omega_{\text{bg}}''(t) = O(1). \quad (3.74)$$

Moreover, direct differentiation of the Bogoliubov dispersion relation gives

$$\frac{\omega_{\text{bg}}'(t)}{2\omega_{\text{bg}}(t)} - \frac{1}{t} = \frac{\nu(t\hat{v}'(t) - 2\hat{v}(t))}{t(\hat{v}(0)t^2 + 4\nu\hat{v}(t))}. \quad (3.75)$$

Consequently,

$$\left| \frac{\omega_{\text{bg}}'(t)}{2\omega_{\text{bg}}(t)} - \frac{1}{t} \right| \leq C \left(\frac{|\hat{v}'(t)|}{t^2} + \frac{|\hat{v}(t)|}{t^3} \right). \quad (3.76)$$

Since $\hat{v}$ and $\hat{v}'$ are bounded under our assumptions, the explicit formula for the derivative of j_+ yields

$$\left| (\partial_p - \partial_q)^2 j_+(p, q) \Big|_{p=q=t} \right| \leq C \left(|\hat{v}''(t)| + \frac{|\hat{v}'(t)|}{t^2} + \frac{|\hat{v}(t)|}{t^4} \right). \quad (3.77)$$

It follows that integral (3.45) is absolutely integrable at infinity.

In Section 6 we justify these expansions proving the following proposition:

Proposition 3.4. *Under Assumptions 2.1 and 3.1, we have*

$$\delta_3^{\text{can}}(\mathbf{k}, \nu, \hat{v}) = -c_{\text{ir}}(k, \nu, \hat{v}) + k c_3(\nu, \hat{v}) + o\left(\frac{k}{\sqrt{\nu}}\right), \quad (3.78)$$

where c_{ir} is given in (3.60) and c_3 in (3.67)-(3.69).

3.3. Real energy shift. In this subsection we present the general second-order expression for the real energy shift. This is obtained by summing the contributions δ_3^{can} and δ_4^{can} from Propositions 3.4 and 3.2.

After summing the two contributions, the infrared-divergent terms $c_{\text{ir}}(\mathbf{k}, \nu, \hat{v})$ and $-c_{\text{ir}}(\mathbf{k}, \nu, \hat{v})$ cancel, leaving a correction whose leading term is linear in $\mathbf{k}$ as $\frac{|\mathbf{k}|}{\sqrt{\nu}} \rightarrow 0$. To simplify the derivation, throughout this subsection we will fix $\nu = \hat{v}(\mathbf{0}) = 1$. The dependence on these parameters will be restored at the end.

Writing

$$s_p^2 = \frac{\frac{p^2}{2} + \hat{v}(p) - \omega_{\text{bg}}(p)}{2\omega_{\text{bg}}(p)}, \quad (3.79)$$

we can rewrite the Hellmann-Feynman correction (3.25)-(3.26) as

$$c_4(1, \hat{v}) = \frac{1}{12\pi^2} \int_0^\infty dp s_p^2 \frac{d}{dp} (p^2 \hat{v}'(p) - p^3) \quad (3.80)$$

$$+ \frac{1}{24\pi^2} \int_0^\infty dp \frac{\hat{v}(p)}{\omega_{\text{bg}}(p)} \frac{d}{dp} (p^2 \hat{v}'(p)). \quad (3.81)$$

According to (3.67)-(3.69) the Fermi Golden Rule contribution to the phonon velocity is

$$c_3(1, \hat{v}) = -\frac{1}{32\pi^2} \int_0^\infty dp \frac{p^6 \hat{v}(p)^2}{\omega_{\text{bg}}(p)^3} + \frac{1}{16\pi^2} \quad (3.82)$$

$$- \frac{1}{16\pi^2} \int_0^\infty dp \left[-\frac{p^2 \hat{v}(p)^2 \omega_{\text{bg}}''(p)}{3\omega_{\text{bg}}(p)^2} + \frac{2p^2 \hat{v}(p)}{3\omega_{\text{bg}}(p)} \left(\hat{v}''(p) - 4\hat{v}'(p) \left(\frac{\omega_{\text{bg}}'(p)}{2\omega_{\text{bg}}(p)} - \frac{1}{p} \right) \right) + \frac{2\hat{v}(p)^2}{3\omega_{\text{bg}}(p)} \left(-1 + 4p^2 \left(\frac{\omega_{\text{bg}}'(p)}{2\omega_{\text{bg}}(p)} - \frac{1}{p} \right)^2 \right) \right]. \quad (3.83)$$

While the above formula might look extremely overwhelming, there are some hidden simplifications that will render it considerably simpler. The integrand in the last two lines can be rewritten as

$$+ \frac{p^2 \omega_{\text{bg}}''(p) \hat{v}(p)^2}{48\omega_{\text{bg}}(p)^2} - \frac{p^2 \hat{v}(p) \hat{v}''(p)}{24\omega_{\text{bg}}(p)} + \frac{p^2 \omega_{\text{bg}}'(p) \hat{v}(p) \hat{v}'(p)}{12\omega_{\text{bg}}(p)^2} - \frac{p \hat{v}'(p) \hat{v}(p)}{6\omega_{\text{bg}}(p)} \quad (3.84)$$

$$- \frac{\omega_{\text{bg}}'(p)^2 \hat{v}(p)^2 p^2}{24\omega_{\text{bg}}(p)^3} + \frac{\omega_{\text{bg}}'(p) \hat{v}(p)^2 p}{6\omega_{\text{bg}}(p)^2} - \frac{\hat{v}(p)^2}{8\omega_{\text{bg}}(p)} \quad (3.85)$$

$$= - \frac{\omega_{\text{bg}}'(p)^2 \hat{v}(p)^2 p^2}{24\omega_{\text{bg}}(p)^3} + \frac{p^2 \omega_{\text{bg}}''(p) \hat{v}(p)^2}{48\omega_{\text{bg}}(p)^2} + \frac{p^2 \omega_{\text{bg}}'(p) \hat{v}(p) \hat{v}'(p)}{24\omega_{\text{bg}}(p)^2} \quad (3.86)$$

$$+ \frac{\hat{v}(p)^2}{24\omega_{\text{bg}}(p)} + \frac{p \hat{v}'(p) \hat{v}(p)}{12\omega_{\text{bg}}(p)} \quad (3.87)$$

$$- \frac{\hat{v}(p)}{24\omega_{\text{bg}}(p)} \left(4\hat{v}(p) + 4p \hat{v}'(p) - \frac{\omega_{\text{bg}}'(p)}{\omega_{\text{bg}}(p)} [p^2 \hat{v}'(p) + 4\hat{v}(p)p] \right) \quad (3.88)$$

$$- \frac{p \hat{v}(p)}{24\omega_{\text{bg}}(p)} (p \hat{v}''(p) + 2\hat{v}'(p)). \quad (3.89)$$

Let us analyze separately the expressions in the four lines (3.86)–(3.89):

$$(3.86) + \frac{p\omega'_{\text{bg}}(p)\hat{v}(p)^2}{24\omega_{\text{bg}}(p)^2} = \frac{d}{dp} \frac{p^2\omega'_{\text{bg}}(p)\hat{v}(p)^2}{48\omega_{\text{bg}}(p)^2}, \quad (3.90)$$

$$(3.87) - \frac{p\omega'_{\text{bg}}(p)\hat{v}(p)^2}{24\omega_{\text{bg}}(p)^2} = \frac{d}{dp} \frac{p\hat{v}(p)^2}{24\omega_{\text{bg}}(p)}. \quad (3.91)$$

Next, using

$$\omega_{\text{bg}}(p)^2 = \frac{1}{4}p^4 + p^2\hat{v}(p), \quad \omega_{\text{bg}}(p)\omega'_{\text{bg}}(p) = \frac{1}{2}p^3 + p\hat{v}(p) + \frac{1}{2}p^2\hat{v}'(p), \quad (3.92)$$

we compute

$$\begin{aligned} (3.88) &= \frac{\hat{v}(p)}{24\omega_{\text{bg}}(p)^3} \left(-4\omega_{\text{bg}}(p)^2(\hat{v}(p) + p\hat{v}'(p)) + \omega_{\text{bg}}(p)\omega'_{\text{bg}}(p)(p^2\hat{v}'(p) + 4p\hat{v}(p)) \right) \\ &= \frac{p^2}{12}(\hat{v}'(p) - p) \frac{p\hat{v}(p)(\hat{v}'(p)p - 2\hat{v}(p))}{4\omega_{\text{bg}}(p)^3} \\ &= \frac{p^2}{12}(\hat{v}'(p) - p) \frac{d}{dp} s_p^2 \end{aligned} \quad (3.93)$$

Finally,

$$(3.89) = -\frac{\hat{v}(p)}{24\omega_{\text{bg}}(p)} \frac{d}{dp} p^2\hat{v}'(p). \quad (3.94)$$

Thus

$$c_3(1, \hat{v}) = -\frac{1}{32\pi^2} \int_0^\infty dp \frac{p^6\hat{v}(p)^2}{\omega_{\text{bg}}(p)^3} + \frac{1}{16\pi^2} \quad (3.95)$$

$$+ \frac{1}{12\pi^2} \int_0^\infty dp (p^2\hat{v}'(p) - p^3) \frac{d}{dp} s_p^2 \quad (3.96)$$

$$- \frac{1}{24\pi^2} \int_0^\infty dp \frac{\hat{v}(p)}{\omega_{\text{bg}}(p)} \frac{d}{dp} (p^2\hat{v}'(p)) \quad (3.97)$$

$$+ \frac{1}{\pi^2} \int_0^\infty dp \left(\frac{d}{dp} \frac{p^2\omega'_{\text{bg}}(p)\hat{v}(p)^2}{48\omega_{\text{bg}}(p)^2} + \frac{d}{dp} \frac{p\hat{v}(p)^2}{24\omega_{\text{bg}}(p)} \right). \quad (3.98)$$

Adding (3.80)–(3.81) and (3.95)–(3.98), and performing the integration of the derivatives we obtain

$$c_3(1, \hat{v}) + c_4(1, \hat{v}) = -\frac{1}{32\pi^2} \int_0^\infty dp \frac{p^6\hat{v}(p)^2}{\omega_{\text{bg}}(p)^3} + \frac{1}{16\pi^2} \quad (3.99)$$

$$+ \frac{1}{8\pi^2} \left(\frac{2}{3} p^2 s_p^2 (p - \hat{v}'(p)) + \frac{p^2\hat{v}(p)^2\omega'_{\text{bg}}(p)}{6\omega_{\text{bg}}(p)^2} + \frac{p\hat{v}(p)^2}{3\omega_{\text{bg}}(p)} \right) \Big|_0^{+\infty}. \quad (3.100)$$

At the origin

$$s_p = O(\sqrt{p}^{-1}), \quad \hat{v}(p) = 1 + O(p), \quad \hat{v}'(p) = O(p), \quad \omega_{\text{bg}}(p) = O(p), \quad \omega'_{\text{bg}}(p) = 1 + O(p).$$

Therefore, we have

$$\lim_{p \rightarrow 0} \frac{1}{8\pi^2} \left[\frac{2}{3} p^2 s_p^2 (p - \hat{v}'(p)) + \frac{p^2\hat{v}(p)^2\omega'_{\text{bg}}(p)}{6\omega_{\text{bg}}(p)^2} + \frac{p\hat{v}(p)^2}{3\omega_{\text{bg}}(p)} \right] = \frac{1}{16\pi^2}, \quad (3.101)$$

which cancels with the positive $\frac{1}{16\pi^2}$ contribution in $c_3 + c_4$. At infinity instead, by $s_p = O(p^{-2})$, $\omega_{\text{bg}}(p) = \frac{p^2}{2} + O(p)$ and $\omega'_{\text{bg}}(p) = O(p)$ we find

$$\lim_{p \rightarrow +\infty} -\frac{1}{8\pi^2} \left[\frac{2}{3} p^2 s_p^2 (p - \hat{v}'(p)) - \frac{p^2\hat{v}(p)^2\omega'_{\text{bg}}(p)}{6\omega_{\text{bg}}(p)^2} - \frac{p\hat{v}(p)^2}{3\omega_{\text{bg}}(p)} \right] = 0. \quad (3.102)$$

We conclude that

$$c_3(1, \hat{v}) + c_4(1, \hat{v}) = -\frac{1}{32\pi^2} \int_0^\infty dp \frac{p^6 \hat{v}(p)^2}{\omega_{\text{bg}}(p)^3}.$$

Finally, rescaling momenta by $\sqrt{\nu}$ and replacing the normalized potential by $\frac{\hat{v}(\sqrt{\nu} \cdot)}{\hat{v}(0)}$ gives

$$\begin{aligned} c_3(\nu, \hat{v}) + c_4(\nu, \hat{v}) &= \nu \hat{v}(0) \left[c_3 \left(1, \frac{\hat{v}(\sqrt{\nu} \cdot)}{\hat{v}(0)} \right) + c_4 \left(1, \frac{\hat{v}(\sqrt{\nu} \cdot)}{\hat{v}(0)} \right) \right] \\ &= -\frac{\sqrt{\nu}}{32\pi^2 \hat{v}(0)} \int_0^\infty dp \frac{p^6 \hat{v}(p)^2}{\omega_{\text{bg}}(p)^3}. \end{aligned} \quad (3.103)$$

Restoring the perturbative factor κ and using $\omega_{\text{bg}}(k) = \sqrt{\nu}k + O(k^3)$ for fixed $\nu > 0$, we have obtained:

Theorem 3.5. *Suppose that the potential satisfies Assumptions 2.1 and 3.1. Then, for fixed $\nu \in]0, \nu_0]$, the Bogoliubov dispersion relation, including the second-order perturbative correction, satisfies*

$$\begin{aligned} \omega_{\text{bg}}(k) + \kappa (\delta_3^{\text{can}}(\mathbf{k}, \nu, \hat{v}) + \delta_4^{\text{can}}(\mathbf{k}, \nu, \hat{v})) &= k c_{\text{can}}(\nu, \hat{v}) + o\left(\frac{k}{\sqrt{\nu}}\right), \\ \text{as } \frac{k}{\sqrt{\nu}} &\rightarrow 0 \end{aligned} \quad (3.104)$$

where the coefficient c_{can} is given by

$$c_{\text{can}}(\nu, \hat{v}) = \sqrt{\nu} - \kappa \hat{v}(0) \sqrt{\nu} \int_0^{+\infty} \frac{dp}{32\pi^2} \frac{\hat{v}(p)^2}{\hat{v}(0)^2} \frac{p^6}{\omega_{\text{bg}}(p)^3} \quad (3.105)$$

3.4. Proof of identity (3.53). Recall that we are fixing $\nu = 1$ and $\hat{v}(0) = 1$

Lemma 3.6. *The identity (3.53) holds.*

Proof. We first compute

$$(\partial_p - \partial_q)^2 j_+(p, q) \Big|_{p=q=t} = 2\hat{v}''(t) + 8\hat{v}'(t)(c'_t s_t - s'_t c_t) - 8((c'_t)^2 - (s'_t)^2)\hat{v}(t) \quad (3.106)$$

This can be expressed explicitly in terms of $\hat{v}(t)$, $\hat{v}'(t)$ and $\omega_{\text{bg}}(t), \omega'_{\text{bg}}(t)$. We write first $(c'_t)^2 - (s'_t)^2$. We have

$$c'_t - s'_t = \frac{d}{dt} \left(\frac{t}{\sqrt{2\omega_{\text{bg}}(t)}} \right) = (c_t - s_t) \left(\frac{1}{t} - \frac{\omega'_{\text{bg}}(t)}{2\omega_{\text{bg}}(t)} \right), \quad (3.107)$$

and

$$c'_t + s'_t = -\frac{c'_t - s'_t}{(c_t - s_t)^2}, \quad (3.108)$$

which imply

$$\begin{aligned} (c'_t)^2 - (s'_t)^2 &= (c'_t - s'_t)(c'_t + s'_t) \\ &= -\left(\frac{c'_t - s'_t}{c_t - s_t} \right)^2 = -\left(\frac{1}{t} - \frac{\omega'_{\text{bg}}(t)}{2\omega_{\text{bg}}(t)} \right)^2. \end{aligned} \quad (3.109)$$

In the last equality above we used Equation (3.107). To derive a useful expression for $c'_t s_t - s'_t c_t$ we write

$$\frac{d}{dt}(c_t^2 - s_t^2) = 2c_t c'_t - 2s_t s'_t = 0, \quad (3.110)$$

which gives us

$$c'_t s_t - s'_t c_t = c'_t s_t - s'_t c_t + c'_t c_t - s'_t s_t = \frac{c_t + s_t}{c_t + s_t} \frac{c'_t - s'_t}{c_t - s_t}. \quad (3.111)$$

Using again Equation (3.107) we arrive at

$$c'_t s_t - s'_t c_t = \left(\frac{1}{t} - \frac{\omega'_{\text{bg}}(t)}{2\omega_{\text{bg}}(t)} \right). \quad (3.112)$$

Thus, we finally arrive at the expression (3.53). $\square$

3.5. Comparison of weak coupling and dilute limits. Equation (3.105) provides the first-order correction in κ to the Bogoliubov dispersion relation. We emphasize that this formula is derived within perturbation theory, in the weak-coupling regime $\hat{v} \rightarrow \kappa \hat{v}$. Our aim is to relate this result, at least at a formal level, to the better known corrections obtained by Beliaev in the dilute regime. In that setting, the relevant parameters are the gas density n and the scattering length $\mathbf{a}$, with the product $n\mathbf{a}^3$ assumed to be small. It is well known that, under suitable assumptions, the scattering length can be expanded as a series in the interaction potential, see [6, 30] and Appendix A:

$$\begin{aligned} 4\pi\mathbf{a} &= \kappa\hat{v}(\mathbf{0}) \\ &+ \sum_{k=1}^{+\infty} (-1)^k \kappa^{k+1} \int \frac{d\mathbf{p}_1 \dots d\mathbf{p}_k}{((2\pi)^3)^k} \frac{\hat{v}(\mathbf{p}_1)\hat{v}(\mathbf{p}_1 - \mathbf{p}_2) \dots \hat{v}(\mathbf{p}_{k-1} - \mathbf{p}_k)\hat{v}(\mathbf{p}_k)}{|\mathbf{p}_1|^2 \dots |\mathbf{p}_k|^2}, \\ &= 4\pi \sum_{k=0}^{\infty} \kappa^{k+1} \mathbf{a}_k, \end{aligned} \quad (3.113)$$

which is convergent for sufficiently regular and small potentials.

The first two terms of this series are

$$\mathbf{a}_0 = \frac{\hat{v}(\mathbf{0})}{4\pi}, \quad \mathbf{a}_1 = -\frac{1}{4\pi} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{\hat{v}(\mathbf{p})^2}{|\mathbf{p}|^2} = -\int_0^{+\infty} \frac{dp}{(2\pi)^3} \hat{v}(p)^2. \quad (3.114)$$

All terms in the Born series beyond $\mathbf{a}_0$ diverge in the formal flat-Fourier limit $\hat{v}(k) = \hat{v}(0)$.

Recall that $\nu = \kappa\hat{v}(\mathbf{0})n$, $\omega_{\text{bg}}(p) = \sqrt{\frac{p^2}{4} + \frac{\hat{v}(p)}{\hat{v}(0)}\nu p^2}$. The second term in c_{can} can be written as

$$- \kappa\hat{v}(0)\nu \int_0^{\infty} \frac{dp}{32\pi^2} \frac{\hat{v}(\sqrt{\nu}p)^2}{\hat{v}(0)^2} \frac{\nu^3 p^6}{\omega_{\text{bg}}(\sqrt{\nu}p)^3} \quad (3.115)$$

$$= - \frac{\kappa\sqrt{\nu}}{\hat{v}(0)} \int_0^{+\infty} \frac{dp}{4\pi^2} \hat{v}(p)^2. \quad (3.116)$$

$$- \kappa\hat{v}(0)\nu \int_0^{+\infty} \frac{dp}{32\pi^2} \left(\frac{p^6}{(\frac{p^4}{4} + p^2)^{\frac{3}{2}}} - 8 \right) \quad (3.117)$$

$$- \kappa\hat{v}(0)\nu \int_0^{\infty} \frac{dp}{32\pi^2} \left(\frac{\hat{v}(\sqrt{\nu}p)^2}{\hat{v}(0)^2} - 1 \right) \left(\frac{\nu^3 p^6}{(\frac{p^4}{4} + p^2)^{\frac{3}{2}}} - 8 \right) \quad (3.118)$$

$$- \kappa\hat{v}(0)\nu \int_0^{\infty} \frac{dp}{32\pi^2} \frac{\hat{v}(\sqrt{\nu}p)^2}{\hat{v}(0)^2} \left(\frac{\nu^3 p^6}{\omega_{\text{bg}}(\sqrt{\nu}p)^3} - \frac{p^6}{(\frac{p^4}{4} + p^2)^{\frac{3}{2}}} \right). \quad (3.119)$$

Now c_{can} consists of the following terms:

$$\sqrt{\nu} = \sqrt{4\pi\kappa\mathfrak{a}_0 n}, \quad (3.120)$$

$$(3.116) = \frac{\sqrt{4\pi n\kappa^2\mathfrak{a}_1}}{2\sqrt{\kappa\mathfrak{a}_0}}, \quad (3.121)$$

$$(3.117) = \frac{\kappa\hat{v}(0)\nu}{\pi^2} = \frac{\sqrt{4\pi n\kappa\mathfrak{a}_0}\sqrt{n(4\pi\kappa\mathfrak{a}_0)^3}}{\pi^2} \quad (3.122)$$

$$= \frac{\sqrt{4\pi n\mathfrak{a}}\sqrt{n(4\pi\mathfrak{a})^3}}{\pi^2} + O(\kappa^{\frac{5}{2}}), \quad (3.123)$$

$$(3.118) = O(\kappa^{\frac{5}{2}}), \quad (3.124)$$

$$(3.119) = O(\kappa^{\frac{5}{2}}). \quad (3.125)$$

Now the first two terms of c_{can} give

$$\sqrt{\nu} + (3.116) = \sqrt{4\pi n} \left(\sqrt{\kappa\mathfrak{a}_0} + \frac{\kappa^2\mathfrak{a}_1}{2\sqrt{\kappa\mathfrak{a}_0}} \right) \quad (3.126)$$

$$= \sqrt{4\pi n} \sqrt{\kappa\mathfrak{a}_0 + \kappa^2\mathfrak{a}_1} + O(\kappa^{\frac{5}{2}}) = \sqrt{4\pi n\mathfrak{a}} + O(\kappa^{\frac{5}{2}}). \quad (3.127)$$

Adding the contribution from (3.117), we obtain

$$c_{\text{can}}(\nu, \hat{v}) = \sqrt{4\pi n\mathfrak{a}} \left(1 + \frac{1}{\pi^2} \sqrt{n(4\pi\mathfrak{a})^3} \right) + O(\kappa^{\frac{5}{2}}). \quad (3.128)$$

Next we express the gas density in terms of the density of the condensate:

$$n = n_{0,\text{bg}} + \frac{\nu^{\frac{3}{2}}}{3\pi^2} = n_{0,\text{bg}} \left(1 + \frac{\sqrt{n_{0,\text{bg}}(4\pi\kappa\mathfrak{a}_0)^3}}{3\pi^2} \right) \quad (3.129)$$

$$= n_{0,\text{bg}} \left(1 + \frac{\sqrt{n_{0,\text{bg}}(4\pi\mathfrak{a})^3}}{3\pi^2} \right) + O(\kappa^{\frac{5}{2}}), \quad (3.130)$$

$$\sqrt{n} = \sqrt{n_{0,\text{bg}}} \left(1 + \frac{\sqrt{n_{0,\text{bg}}(4\pi\mathfrak{a})^3}}{6\pi^2} \right) + O(\kappa^{\frac{5}{2}}). \quad (3.131)$$

Thus we can recast the correction as

$$c_{\text{can}}(\nu, \hat{v}) = \sqrt{4\pi n_{0,\text{bg}}\mathfrak{a}} \left(1 + \frac{7}{6\pi^2} \sqrt{n_{0,\text{bg}}(4\pi\mathfrak{a})^3} \right) + O(\kappa^{\frac{5}{2}}), \quad (3.132)$$

which is consistent with Beliaev's original result [2]. This essentially proves Corollary 1.3

4. GRAND-CANONICAL DERIVATION

Let us describe an alternative approach to the Bose gas based on the grand-canonical formalism. We will see that it yields equivalent description of the shift of the quasiparticle spectrum as the canonical approach, which we applied so far in our paper.

Let H^L be the “first-principles Hamiltonian” (2.10), where we make the coupling constant κ explicit. It acts on the full Fock space $\Gamma_{\text{s}}(\ell^2(\Xi_L))$, which includes the zero-mode. Introduce the grand-canonical Hamiltonian

$$H_{\mu}^L := H^L - \mu N^L, \quad (4.1)$$

where μ is the *grand canonical chemical potential*. We expect that at zero temperature the physical system is described by the ground state of H_{μ}^L and its low-lying states.

Again, we will view H_μ^L as a perturbation of an appropriate quadratic Hamiltonian, whose choice we are now going to describe. First for $\alpha \geq 0$ we introduce the Weyl operator of the mode $\mathbf{k} = 0$

$$W_\alpha = \exp(-\alpha a_0^* + \bar{\alpha} a_0). \quad (4.2)$$

We set $\Omega_\alpha = W_\alpha^* \Omega$. Instead of α in the sequel we will often prefer the parameter $\mu_0 \in \mathbb{R}_+$ so that

$$\alpha = \sqrt{\frac{\mu_0 L^d}{\kappa \hat{v}(\mathbf{0})}}. \quad (4.3)$$

We compute

$$(\Omega_\alpha | a_0 \Omega_\alpha) = (\Omega_\alpha | a_0^* \Omega_\alpha) = \frac{\sqrt{\mu_0 L^d}}{\sqrt{\kappa \hat{v}(\mathbf{0})}}, \quad (4.4)$$

$$(\Omega_\alpha | a_0 a_0 \Omega_\alpha) = (\Omega_\alpha | a_0^* a_0^* \Omega_\alpha) = (\Omega_\alpha | a_0^* a_0 \Omega_\alpha) = \frac{\mu_0 L^d}{\kappa \hat{v}(\mathbf{0})}, \quad (4.5)$$

$$\begin{aligned} (\Omega_\alpha | (H^L - \mu N^L) \Omega_\alpha) &= -\mu |\alpha|^2 + \frac{\kappa \hat{v}(\mathbf{0})}{2L^d} |\alpha|^4 \\ &= \frac{L^d}{\kappa \hat{v}(\mathbf{0})} \left(-\mu \mu_0 + \frac{\mu_0^2}{2} \right). \end{aligned} \quad (4.6)$$

By (4.5), μ_0 is responsible for the density of the condensate.

One possibility, see e.g. [21], consists in choosing the quadratic Hamiltonian determined by $\alpha = \sqrt{\frac{\mu L^d}{\kappa \hat{v}(\mathbf{0})}}$, so that Ω_α minimizes (4.6). This would mean $\mu = \mu_0$. However, we prefer to make a better choice of μ_0 , which will be different from μ . For a while, we keep μ_0 as an additional free parameter, which will be fixed soon.

It is convenient to introduce new annihilation operators

$$W_\alpha^* a_{\mathbf{k}} W_\alpha = a_{\mathbf{k}} - \alpha \delta_{\mathbf{k},0} =: \tilde{a}_{\mathbf{k}}. \quad (4.7)$$

They kill the new vacuum Ω_α . Omitting the tildes, we obtain the Hamiltonian translated by the transformation (4.7):

$$\begin{aligned} H^L - \mu N^L &= H_{0,1}^L + H_{0,2}^L + H_{0,3}^L + H_{0,4}^L + E_0^L + H_{\text{bg},\mu_0}^L + \sqrt{\kappa} H_{3,\mu_0}^L + \kappa H_{4,\mu_0}^L \\ &\quad - (\mu - \mu_0) N^>, \\ E_0^L &:= \frac{L^d}{\kappa \hat{v}(\mathbf{0})} \left(\frac{\mu_0^2}{2} - \mu_0 \mu \right), \\ H_{\text{bg},\mu_0}^L &:= \sum_{\mathbf{p} \in \Xi_L^>} \left(\frac{\mathbf{p}^2}{2} + \frac{\mu_0 \hat{v}(\mathbf{p})}{\hat{v}(\mathbf{0})} \right) a_{\mathbf{p}}^* a_{\mathbf{p}} + \sum_{\mathbf{p} \in \Xi_L^>} \frac{\mu_0 \hat{v}(\mathbf{p})}{2\hat{v}(\mathbf{0})} (a_{\mathbf{p}}^* a_{-\mathbf{p}}^* + a_{\mathbf{p}} a_{-\mathbf{p}}), \\ H_{3,\mu_0}^L &:= \frac{1}{L^{\frac{d}{2}}} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{p}+\mathbf{q} \in \Xi_L^>} \frac{\hat{v}(\mathbf{p}) \sqrt{\mu_0}}{\sqrt{\hat{v}(\mathbf{0})}} (a_{\mathbf{p}+\mathbf{q}}^* a_{\mathbf{p}} a_{\mathbf{q}} + a_{\mathbf{p}}^* a_{\mathbf{q}}^* a_{\mathbf{p}+\mathbf{q}}), \\ H_{4,\mu_0}^L &:= \frac{1}{2L^d} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{p}+\mathbf{k}, \mathbf{q}-\mathbf{k} \in \Xi_L^>} \hat{v}(\mathbf{k}) a_{\mathbf{p}+\mathbf{k}}^* a_{\mathbf{q}-\mathbf{k}}^* a_{\mathbf{q}} a_{\mathbf{p}}. \end{aligned} \quad (4.8)$$

The operators $H_{0,1}^L, H_{0,2}^L, H_{0,3}^L$, and $H_{0,4}^L$ (which we do not write explicitly) contain respectively products of two, three and four annihilation and creation operators, always including at least one normally ordered a_0 or a_0^* .

Following the intuition in [35], we will start from μ_0 minimizing approximately the ground state energy of H_μ^L . More precisely, we will tune μ_0 so as to minimize the expectation value of H_μ^L in $\Omega_{\text{bg}}(\mu_0)$, the ground state of the Bogoliubov Hamiltonian associated with the parameter μ_0 . Differently from the canonical approach, the

state $\Omega_{\text{bg}}(\mu_0)$ depends now on μ_0 , whereas the canonical state was constructed starting from ν . This state can be written in terms of the shifted vacuum Ω_α as

$$\Omega_{\text{bg}}(\mu_0) = \prod_{\mathbf{p} \in \Xi_L^{\geq}} \frac{1}{(\cosh \theta_{\mathbf{p}})^{\frac{1}{2}}} e^{-\frac{1}{2} \tanh \theta_{\mathbf{p}} a_{\mathbf{p}}^* a_{-\mathbf{p}}} \Omega_\alpha. \quad (4.9)$$

Now,

$$(\Omega_{\text{bg}}(\mu_0) | H_\mu^L \Omega_{\text{bg}}(\mu_0)) = \frac{L^d}{\kappa \hat{v}(\mathbf{0})} \left(\frac{\mu_0^2}{2} - \mu \mu_0 \right) + (\Omega_{\text{bg}}(\mu_0) | H_{\text{bg}, \mu_0}^L \Omega_{\text{bg}}(\mu_0)) \quad (4.10)$$

$$- (\mu - \mu_0) (\Omega_{\text{bg}}(\mu_0) | N^> \Omega_{\text{bg}}(\mu_0)) + \kappa (\Omega_{\text{bg}}(\mu_0) | H_{4, \mu_0}^L \Omega_{\text{bg}}(\mu_0)). \quad (4.11)$$

Indeed, the expectation values of the terms containing zero-mode operators and of the cubic term vanish in $\Omega_{\text{bg}}(\mu_0)$.

Recall from (2.65) that

$$(\Omega_{\text{bg}}(\mu_0) | H_{\text{bg}, \mu_0}^L \Omega_{\text{bg}}(\mu_0)) = E_{\text{bg}}^L(\mu_0) \quad (4.12)$$

$$= - \sum_{\mathbf{p} \in \Xi_L^{\geq}} \frac{1}{2} \left(\frac{1}{2} \mathbf{p}^2 + \mu_0 \frac{\hat{v}(\mathbf{p})}{\hat{v}(\mathbf{0})} - \omega_{\text{bg}}(\mathbf{p}, \mu_0) \right). \quad (4.13)$$

Therefore,

$$\frac{\partial}{\partial \mu_0} (\Omega_{\text{bg}}(\mu_0) | H_{\text{bg}, \mu_0}^L \Omega_{\text{bg}}(\mu_0)) = \sum_{\mathbf{p} \in \Xi_L^{\geq}} \frac{\hat{v}(\mathbf{p}) (\frac{\mathbf{p}^2}{2} - \omega_{\text{bg}}(\mathbf{p}, \mu_0))}{\hat{v}(\mathbf{0}) 2 \omega_{\text{bg}}(\mathbf{p}, \mu_0)} \quad (4.14)$$

$$= \sum_{\mathbf{p} \in \Xi_L^{\geq}} \frac{\hat{v}(\mathbf{p})}{\hat{v}(\mathbf{0})} (s_{\mathbf{p}}^2 - c_{\mathbf{p}} s_{\mathbf{p}}). \quad (4.15)$$

Moreover,

$$(\Omega_{\text{bg}}(\mu_0) | N^> \Omega_{\text{bg}}(\mu_0)) = \sum_{\mathbf{p} \in \Xi_L^{\geq}} s_{\mathbf{p}}^2. \quad (4.16)$$

Thus

$$0 = \frac{\partial}{\partial \mu_0} (\Omega_{\text{bg}}(\mu_0) | H_\mu^L \Omega_{\text{bg}}(\mu_0)) \quad (4.17)$$

$$= \frac{L^d}{\kappa \hat{v}(\mathbf{0})} (\mu_0 - \mu) + \sum_{\mathbf{p} \in \Xi_L^{\geq}} \left[\left(1 + \frac{\hat{v}(\mathbf{p})}{\hat{v}(\mathbf{0})} \right) s_{\mathbf{p}}^2 - \frac{\hat{v}(\mathbf{p})}{\hat{v}(\mathbf{0})} c_{\mathbf{p}} s_{\mathbf{p}} \right] \quad (4.18)$$

$$+ (\mu_0 - \mu) \frac{\partial}{\partial \mu_0} \sum_{\mathbf{p} \in \Xi_L^{\geq}} s_{\mathbf{p}}^2 + \kappa \frac{\partial}{\partial \mu_0} (\Omega_{\text{bg}}(\mu_0) | H_{4, \mu_0}^L \Omega_{\text{bg}}(\mu_0)). \quad (4.19)$$

Multiplying the stationarity equation by $\frac{\kappa \hat{v}(\mathbf{0})}{L^d}$, we obtain

$$\mu_0 - \mu = - \frac{\kappa \hat{v}(\mathbf{0})}{L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} \left[\left(1 + \frac{\hat{v}(\mathbf{p})}{\hat{v}(\mathbf{0})} \right) s_{\mathbf{p}}^2 - \frac{\hat{v}(\mathbf{p})}{\hat{v}(\mathbf{0})} c_{\mathbf{p}} s_{\mathbf{p}} \right] \quad (4.20)$$

$$- \frac{\kappa \hat{v}(\mathbf{0}) (\mu_0 - \mu)}{L^d} \frac{\partial}{\partial \mu_0} \sum_{\mathbf{p} \in \Xi_L^{\geq}} s_{\mathbf{p}}^2 - \frac{\kappa^2 \hat{v}(\mathbf{0})}{L^d} \frac{\partial}{\partial \mu_0} (\Omega_{\text{bg}}(\mu_0) | H_{4, \mu_0}^L \Omega_{\text{bg}}(\mu_0)). \quad (4.21)$$

The term involving H_4^L carries an additional factor of κ^2 . Therefore, we will drop it. The first term in the line (4.21) carries κ and $\mu - \mu_0$, which we know to be of the order κ . Therefore, we will drop it as well.

This justifies setting

$$\begin{aligned}\mu_0 &= \mu - \kappa\mu_1, \\ \mu_1 &:= \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} \left[(\hat{v}(\mathbf{0}) + \hat{v}(\mathbf{p})) s_{\mathbf{p}}^2 - \hat{v}(\mathbf{p}) c_{\mathbf{p}} s_{\mathbf{p}} \right],\end{aligned}\quad (4.22)$$

where $s_{\mathbf{p}}$, $c_{\mathbf{p}}$ are evaluated for μ_0 . (Actually, since $\mu_0 - \mu = O(\kappa)$, $c_{\mathbf{p}}$ and $s_{\mathbf{p}}$ in the definition of μ_1 can be evaluated at μ up to errors $O(\kappa^2)$).

4.1. Grand-canonical energy shift. We can now compute the dispersion-relation shift in the grand-canonical formalism. To this end, in the formula derived for the canonical shift, we replace the canonical Hamiltonian by the corresponding grand-canonical Hamiltonian, with μ_0 chosen as above.

$$\delta^{\text{gcan}}(\mathbf{k}, \mu, \hat{v}) = \delta_4^{\text{gcan}}(\mathbf{k}, \mu, \hat{v}) + \delta_3^{\text{gcan}}(\mathbf{k}, \mu, \hat{v}), \quad (4.23)$$

with

$$\delta_4^{\text{gcan}}(\mathbf{k}, \mu, \hat{v}) := \lim_{L \rightarrow \infty} (\Omega_{\text{bg}}(\mu_0) | [b_{\mathbf{k}}, [H_{4,\mu_0}^L - \mu_1 N^>, b_{\mathbf{k}}^*]] \Omega_{\text{bg}}(\mu_0)), \quad (4.24)$$

and

$$\begin{aligned}\delta_3^{\text{gcan}}(\mathbf{k}, \mu, \hat{v}) &:= \lim_{\epsilon \searrow 0} \lim_{L \rightarrow \infty} \left\{ - (\Omega_{\text{bg}}(\mu_0) | [b_{\mathbf{k}}, H_{3,\mu_0}^L] (H_{\text{bg},\mu_0}^L - \omega_{\text{bg}}(\mathbf{k}, \mu_0) - i\epsilon)^{-1} [H_{3,\mu_0}^L, b_{\mathbf{k}}^*] \Omega_{\text{bg}}(\mu_0)) \right. \\ &\quad - (\Omega_{\text{bg}}(\mu_0) | [b_{\mathbf{k}}^*, H_{3,\mu_0}^L] (H_{\text{bg},\mu_0}^L + \omega_{\text{bg}}(\mathbf{k}, \mu_0) - i\epsilon)^{-1} [H_{3,\mu_0}^L, b_{\mathbf{k}}] \Omega_{\text{bg}}(\mu_0)) \\ &\quad - (\Omega_{\text{bg}}(\mu_0) | H_{3,\mu_0}^L (H_{\text{bg},\mu_0}^L - i\epsilon)^{-1} [b_{\mathbf{k}}, [H_{3,\mu_0}^L, b_{\mathbf{k}}^*]] \Omega_{\text{bg}}(\mu_0)) \\ &\quad \left. - (\Omega_{\text{bg}}(\mu_0) | [b_{\mathbf{k}}^*, [H_{3,\mu_0}^L, b_{\mathbf{k}}]] (H_{\text{bg},\mu_0}^L - i\epsilon)^{-1} H_{3,\mu_0}^L \Omega_{\text{bg}}(\mu_0)) \right\}. \end{aligned} \quad (4.25)$$

The Fermi golden rule contribution is exactly the same in the two formalisms after setting $\mu_0 = \mu = \nu$ (which is true after discarding terms of order κ).

Let us compute the Hellman–Feynman correction. An easy computation gives

$$(\Omega_{\text{bg}}(\mu_0) | [b_{\mathbf{k}}, [H_{4,\mu_0}^L, b_{\mathbf{k}}^*]] \Omega_{\text{bg}}(\mu_0)) \quad (4.26)$$

$$\begin{aligned}&= (c_{\mathbf{k}}^2 + s_{\mathbf{k}}^2) \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} s_{\mathbf{p}}^2 \left(\hat{v}(\mathbf{0}) + \frac{\hat{v}(\mathbf{p} - \mathbf{k}) + \hat{v}(\mathbf{p} + \mathbf{k})}{2} \right) \\ &\quad + 2c_{\mathbf{k}} s_{\mathbf{k}} \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} c_{\mathbf{p}} s_{\mathbf{p}} \left(\frac{\hat{v}(\mathbf{p} - \mathbf{k}) + \hat{v}(\mathbf{p} + \mathbf{k})}{2} \right) + O(L^{-d}).\end{aligned} \quad (4.27)$$

Even after including the contribution of the Fermi Golden Rule, this expression remains infrared divergent (in the thermodynamic limit) as $\mathbf{k} \rightarrow \mathbf{0}$. However, we still have to add the counterterm involving

$$\mu_1 = \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} ((\hat{v}(\mathbf{0}) + \hat{v}(\mathbf{p})) s_{\mathbf{p}}^2 - \hat{v}(\mathbf{p}) c_{\mathbf{p}} s_{\mathbf{p}}), \quad (4.28)$$

$$(\Omega_{\text{bg}}(\mu_0) | [b_{\mathbf{k}}, [N^>, b_{\mathbf{k}}^*]] \Omega_{\text{bg}}(\mu_0)) = (c_{\mathbf{k}}^2 + s_{\mathbf{k}}^2). \quad (4.29)$$

The full expression for the grand-canonical Hellmann-Feynman correction is

$$\delta_4^{\text{gcan},L}(\mathbf{k}, \mu, \hat{v}) = (\Omega_{\text{bg}}(\mu_0) | [b_{\mathbf{k}}, [H_{4,\mu_0}^L - \kappa\mu_1 N^>, b_{\mathbf{k}}^*]] \Omega_{\text{bg}}(\mu_0)) \upharpoonright_{\mu_0=\nu} \quad (4.30)$$

$$= (c_{\mathbf{k}} + s_{\mathbf{k}})^2 \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} c_{\mathbf{p}} s_{\mathbf{p}} \hat{v}(\mathbf{p}) \quad (4.31)$$

$$+ (c_{\mathbf{k}}^2 + s_{\mathbf{k}}^2) \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} s_{\mathbf{p}}^2 \left(-\hat{v}(\mathbf{p}) + \frac{\hat{v}(\mathbf{p} + \mathbf{k}) + \hat{v}(\mathbf{p} - \mathbf{k})}{2} \right) \quad (4.32)$$

$$+ 2c_{\mathbf{k}} s_{\mathbf{k}} \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} c_{\mathbf{p}} s_{\mathbf{p}} \left(-\hat{v}(\mathbf{p}) + \frac{\hat{v}(\mathbf{p} + \mathbf{k}) + \hat{v}(\mathbf{p} - \mathbf{k})}{2} \right) + O(L^{-d}). \quad (4.33)$$

4.2. Comparison of the grand-canonical and canonical approaches. Let us compare (4.30) with the canonical expression for the Hellmann-Feynman term

$$\delta_4^{\text{can},L}(\mathbf{k}, \nu, \hat{v}) := (\Omega_{\text{bg}}(\nu) | [b_{\mathbf{k}}, [H_{4,2}^L, b_{\mathbf{k}}^*]] \Omega_{\text{bg}}(\nu)) =$$

$$= (c_{\mathbf{k}} + s_{\mathbf{k}})^2 \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} c_{\mathbf{p}} s_{\mathbf{p}} \hat{v}(\mathbf{p}) \quad (4.34)$$

$$+ (c_{\mathbf{k}}^2 + s_{\mathbf{k}}^2) \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} s_{\mathbf{p}}^2 \left(-\hat{v}(\mathbf{p}) + \frac{\hat{v}(\mathbf{p} + \mathbf{k}) + \hat{v}(\mathbf{p} - \mathbf{k})}{2} \right) \quad (4.35)$$

$$+ 2c_{\mathbf{k}} s_{\mathbf{k}} \frac{1}{L^d} \sum_{\mathbf{p}, \mathbf{p} \pm \mathbf{k} \in \Xi_L^>} c_{\mathbf{p}} s_{\mathbf{p}} \left(-\hat{v}(\mathbf{p}) + \frac{\hat{v}(\mathbf{p} + \mathbf{k}) + \hat{v}(\mathbf{p} - \mathbf{k})}{2} \right) \quad (4.36)$$

$$- (c_{\mathbf{k}} - s_{\mathbf{k}})^2 \hat{v}(\mathbf{k}) \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} s_{\mathbf{p}}^2 + O(L^{-d}), \quad (4.37)$$

where the above quantities depend on the parameter ν . If we naively set $\nu = \mu_0$, we see that the two expressions differ by the last line in Equation (4.37), which is of the order $O(\kappa)$. Their difference is

$$\delta_4^{\text{can},L}(\mathbf{k}, \nu, \hat{v}) - \delta_4^{\text{gcan},L}(\mathbf{k}, \mu, \hat{v}) \quad (4.38)$$

$$= -\hat{v}(\mathbf{k}) (c_{\mathbf{k}} - s_{\mathbf{k}})^2 \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} s_{\mathbf{p}}^2 + O(L^{-d}) + O(\kappa). \quad (4.39)$$

Let us explain this discrepancy. As we saw, the chemical potential μ and the parameter of the corresponding Bogoliubov Hamiltonian μ_0 differ by a term of order κ . They both differ by a quantity of order κ from the canonical parameter ν . To determine the relationship between the grand-canonical and canonical parameters, let us use the grand-canonical Bogoliubov ground state $\Omega_{\text{bg}}(\mu_0)$:

$$\nu = \kappa \hat{v}(\mathbf{0}) \frac{N}{L^d} = \frac{\kappa \hat{v}(\mathbf{0})}{L^d} \left((\Omega_{\text{bg}}(\mu_0) | N_{\mathbf{0}} \Omega_{\text{bg}}(\mu_0)) + (\Omega_{\text{bg}}(\mu_0) | N^> \Omega_{\text{bg}}(\mu_0)) \right) \quad (4.40)$$

$$= \mu_0 + \frac{\kappa \hat{v}(\mathbf{0})}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} s_{\mathbf{p}}^2 \quad (4.41)$$

(Here $N_{\mathbf{0}}$ is the unshifted number operator). Equivalently,

$$\mu = \nu + \kappa \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} \left[\hat{v}(\mathbf{p}) s_{\mathbf{p}}^2 - \hat{v}(\mathbf{p}) c_{\mathbf{p}} s_{\mathbf{p}} \right]. \quad (4.42)$$

Therefore the unperturbed dispersion relation in the canonical formalism differs from the relation in the grand-canonical formalism by a term of order $O(\kappa)$: both

are given by the same functional form as in $\omega_{\text{bg}}(\mathbf{k}, \mu_0)$, but the former involves ν and the latter depends on μ_0 . Expanding in the Taylor series we have

$$\omega_{\text{bg}}(\mathbf{k}, \nu) = \omega_{\text{bg}}(\mathbf{k}, \mu_0) + (\mu_0 - \nu) \frac{\partial}{\partial \mu_0} \omega_{\text{bg}}(\mathbf{k}, \mu_0) \big|_{\mu_0=\nu} + O(\kappa^2). \quad (4.43)$$

Now

$$\mu_0 - \nu = -\frac{\kappa \hat{v}(\mathbf{0})}{L^d} \sum_{\mathbf{p} \in \Xi_L} s_{\mathbf{p}}^2, \quad (4.44)$$

$$\begin{aligned} \frac{\partial}{\partial \mu_0} \omega_{\text{bg}}(\mathbf{k}, \mu_0) \big|_{\mu_0=\nu} &= \frac{\hat{v}(\mathbf{k}) |\mathbf{k}|^2}{2 \hat{v}(\mathbf{0}) \omega_{\text{bg}}(\mathbf{k})} \\ c_{\mathbf{k}} - s_{\mathbf{k}} &= \frac{|\mathbf{k}|}{\sqrt{2 \omega_{\text{bg}}(\mathbf{k})}}. \end{aligned} \quad (4.45)$$

Hence,

$$(\mu_0 - \nu) \frac{\partial}{\partial \mu_0} \omega_{\text{bg}}(\mathbf{k}, \mu_0) \big|_{\mu_0=\nu} = -\kappa \hat{v}(\mathbf{k}) (c_{\mathbf{k}} - s_{\mathbf{k}})^2 \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} s_{\mathbf{p}}^2. \quad (4.46)$$

We see that the correction in Equation (4.46) exactly coincides with the difference between the Hellmann-Feynman corrections computed in the canonical and grand-canonical formalisms (4.39). Thus, one has

$$\kappa \delta_4^{\text{gcan}}(\mathbf{k}, \mu, \hat{v}) + (\mu_0 - \nu) \frac{\partial}{\partial \mu_0} \omega_{\text{bg}}(\mathbf{k}, \mu_0) \big|_{\mu_0=\nu} = \kappa \delta_4^{\text{can}}(\mathbf{k}, \nu, \hat{v}) + O(\kappa^2) \quad (4.47)$$

Since the Fermi golden rule corrections already essentially coincide, we conclude that

$$\kappa \delta_4^{\text{gcan}}(\mathbf{k}, \mu, \hat{v}) + (\mu_0 - \nu) \frac{\partial}{\partial \mu_0} \omega_{\text{bg}}(\mathbf{k}, \mu_0) \big|_{\mu_0=\nu} = \kappa \delta_4^{\text{can}}(\mathbf{k}, \nu, \hat{v}) + O(\kappa^2) \quad (4.48)$$

which shows the consistency of the grand-canonical and canonical approaches.

If we compare with the canonical approach, the present derivation is, in a sense, cleaner because we do not have to deal with dirty “tail terms”. However, the parameter μ is perhaps sometimes less transparent. Indeed, one needs to renormalize the “effective chemical potential” μ_0 at every step of the perturbative expansion.

5. TAIL ESTIMATES

5.1. Localization of the Bogoliubov vacuum. The terms $R_{4,1}$ and $R_{4,2}$ arise when passing from the physical Hilbert space to the extended Hilbert space. We call them *tail terms* and show that their contributions vanish in the thermodynamic limit. The main ingredient is the exponential concentration of N_0^{ext} around $n_{0,\text{bg}}(L)L^d$ in the Bogoliubov ground state.

This section contains the technical estimates needed to prove Theorem 2.4. Readers primarily interested in the perturbative computation may take that theorem for granted and skip this section.

We begin with an estimate on the fourth powers of $s_{\mathbf{p}}$.

Lemma 5.1. *Fix $d = 2, 3$ and assume Assumption 2.1. Then*

$$\sum_{\mathbf{p} \in \Xi_L^{\geq, +}} s_{\mathbf{p}}^4 = \begin{cases} O(L^2 \log L), & d = 2, \\ O(L^3), & d = 3, \end{cases} \quad L \longrightarrow +\infty. \quad (5.1)$$

Proof. Under Assumption 2.1, the small-momentum asymptotics, continuity, strict stability and large-momentum behaviour imply

$$\begin{aligned} s_{\mathbf{p}}^4 &\leq \frac{C}{|\mathbf{p}|^2} && \text{for } 0 < |\mathbf{p}| \leq 2\pi, \\ s_{\mathbf{p}}^4 &\leq C s_{\mathbf{p}}^2 && \text{for } |\mathbf{p}| > 2\pi, \end{aligned}$$

with $C > 0$ independent of L . Therefore,

$$\sum_{\mathbf{p} \in \Xi_L^{>,+}} s_{\mathbf{p}}^4 \leq C \sum_{\substack{\mathbf{p} \in \Xi_L^{>} \\ |\mathbf{p}| > 2\pi}} s_{\mathbf{p}}^2 + C \sum_{\substack{\mathbf{p} \in \Xi_L^{>} \\ 0 < |\mathbf{p}| \leq 2\pi}} \frac{1}{|\mathbf{p}|^2} \leq C L^d n_{\text{bg}}^{>}(L) + C' L^2 \sum_{n=1}^{[L]} n^{d-3}, \quad (5.2)$$

where $C' > 0$ is also independent of L . The remaining sum is $O(\log L)$ for $d = 2$ and $O(L)$ for $d = 3$, proving (5.1). $\square$

Next, we derive an exponential Markov bound.

Lemma 5.2. *Let A be a self-adjoint operator and Ω a vector of norm 1 such that the quadratic-form expectations*

$$e^{c_1} := (\Omega | e^A \Omega), \quad (5.3)$$

$$e^{c_2} := (\Omega | e^{-A} \Omega) \quad (5.4)$$

are finite. If $I_\lambda := [a - \lambda, a + \lambda]$, with $a \in \mathbb{R}$ and $\lambda > 0$, then

$$(\Omega | \mathbb{1}_{\mathbb{R} \setminus I_\lambda}(A) \Omega) \leq e^{-\lambda - a + c_1} + e^{-\lambda + a + c_2}. \quad (5.5)$$

Proof. Integrate the pointwise inequality

$$\mathbb{1}_{\mathbb{R} \setminus I_\lambda}(x) \leq e^{-\lambda} (e^{x-a} + e^{-(x-a)}) \quad (5.6)$$

against the spectral measure of A in Ω . $\square$

We apply this bound to $N^{>}$.

Lemma 5.3. *Fix $d = 2, 3$ and assume Assumption 2.1. Let $m > 0$ and*

$$I_m = [n_{\text{bg}}^{>}(L) L^d - m L^d, n_{\text{bg}}^{>}(L) L^d + m L^d]. \quad (5.7)$$

For every fixed $\delta \in]0, 1[$ and all sufficiently large L ,

$$(\Omega_{\text{bg}} | \mathbb{1}_{\mathbb{R} \setminus I_m}(N^{>}) \Omega_{\text{bg}}) \leq \exp \left\{ -\frac{\pi(1+\delta)m}{\sqrt{\nu}} L^{d-1} \right\}. \quad (5.8)$$

Proof. Fix $\delta' \in]\delta, 1[$. We apply Lemma 5.2 with $A = \mu N^{>}$, where $\mu > 0$ is chosen so that

$$e^{2\mu} - 1 = \frac{1 + \delta'}{2} \left(\sup_{\mathbf{p} \in \Xi_L^{>}} s_{\mathbf{p}}^2 \right)^{-1}. \quad (5.9)$$

The uniform small-momentum expansion

$$s_{\mathbf{p}}^2 = \frac{\sqrt{\nu}}{2|\mathbf{p}|} + o(|\mathbf{p}|^{-1}), \quad \mathbf{p} \longrightarrow 0, \quad (5.10)$$

together with boundedness of $s_{\mathbf{p}}^2$ outside any fixed neighbourhood of the origin, gives

$$\sup_{\mathbf{p} \in \Xi_L^{>}} s_{\mathbf{p}}^2 = \frac{\sqrt{\nu}}{4\pi} L + o(L), \quad (5.11)$$

Consequently,

$$\mu = \frac{\pi(1+\delta')}{\sqrt{\nu}L} + o(L^{-1}), \quad 0 \leq |(e^{\pm 2\mu} - 1)s_{\mathbf{p}}^2| \leq \frac{1+\delta'}{2} < 1. \quad (5.12)$$

For the negative sign, we also used $1 - e^{-2\mu} \leq e^{2\mu} - 1$.

By (2.80),

$$\begin{aligned} \log(\Omega_{\text{bg}} | e^{\pm\mu(N^> - n_{\text{bg}}^>(L)L^d)} \Omega_{\text{bg}}) \\ = - \sum_{\mathbf{p} \in \Xi_L^{>,+}} \log(1 - (e^{\pm 2\mu} - 1)s_{\mathbf{p}}^2) \mp \mu n_{\text{bg}}^>(L)L^d. \end{aligned} \quad (5.13)$$

For $|x| \leq \frac{1+\delta'}{2}$, we have

$$0 \leq -\log(1-x) - x = x^2 \int_0^1 \frac{v}{1-vx} dv \leq \frac{x^2}{1-\delta'}.$$

Applying this inequality term by term in (5.13), and using $2 \sum_{\mathbf{p} \in \Xi_L^{>,+}} s_{\mathbf{p}}^2 = L^d n_{\text{bg}}^>(L)$, yields

$$\log(\Omega_{\text{bg}} | e^{\pm\mu(N^> - n_{\text{bg}}^>(L)L^d)} \Omega_{\text{bg}}) \leq \frac{e^{\pm 2\mu} - 1 \mp 2\mu}{2} L^d n_{\text{bg}}^>(L) + \frac{(e^{\pm 2\mu} - 1)^2}{1 - \delta'} \sum_{\mathbf{p} \in \Xi_L^{>,+}} s_{\mathbf{p}}^4.$$

Lemma 5.1 gives

$$\log(\Omega_{\text{bg}} | e^{\pm\mu(N^> - n_{\text{bg}}^>(L)L^d)} \Omega_{\text{bg}}) = \begin{cases} O(\log L), & d = 2, \\ O(L), & d = 3, \end{cases} = o(L^{d-1}). \quad (5.14)$$

In (5.5), take $A = \mu N^>$, $a = \mu n_{\text{bg}}^>(L)L^d$ and $\lambda = \mu m L^d$. The preceding estimate gives $c_1 - a = o(L^{d-1})$ and $c_2 + a = o(L^{d-1})$. Hence

$$(\Omega_{\text{bg}} | \mathbb{1}_{\mathbb{R} \setminus I_m}(N^>) \Omega_{\text{bg}}) \leq 2 \exp \left\{ -\frac{\pi(1+\delta')m}{\sqrt{\nu}} L^{d-1} + o(L^{d-1}) \right\}$$

for all sufficiently large L . After replacing δ' with $\delta \in]0, 1[$, $\delta' > \delta$ we can absorb both the $o(L^{d-1})$ remainder and the factor 2. $\square$

The identity $N_{\mathbf{0}}^{\text{ext}} = N - N^>$ yields the corresponding estimates for the zero-mode.

Corollary 5.4. *Fix $d = 2, 3$ and assume Assumption 2.1.*

a) *Suppose that Assumption 2.3 holds. Let $m_0 > 0$ and*

$$I_{m_0} = [n_{0,\text{bg}}(L)L^d - m_0 L^d, n_{0,\text{bg}}(L)L^d + m_0 L^d]. \quad (5.15)$$

For every fixed $\delta \in]0, 1[$ and all sufficiently large L ,

$$(\Omega_{\text{bg}} | \mathbb{1}_{\mathbb{R} \setminus I_{m_0}}(N_{\mathbf{0}}^{\text{ext}}) \Omega_{\text{bg}}) \leq \exp \left\{ -\frac{\pi(1+\delta)m_0}{\sqrt{\nu}} L^{d-1} \right\}. \quad (5.16)$$

b) *If $n_{0,\text{bg}} < 0$, then*

$$\lim_{L \rightarrow +\infty} (\Omega_{\text{bg}} | \mathbb{1}_{]-\infty, 0[}(N_{\mathbf{0}}^{\text{ext}}) \Omega_{\text{bg}}) = 1. \quad (5.17)$$

Proof. a) Since $n_{0,\text{bg}}(L)L^d = N - n_{\text{bg}}^>(L)L^d$, functional calculus gives

$$(\Omega_{\text{bg}} | \mathbb{1}_{\mathbb{R} \setminus I_{m_0}}(N_{\mathbf{0}}^{\text{ext}}) \Omega_{\text{bg}}) = (\Omega_{\text{bg}} | \mathbb{1}_{\mathbb{R} \setminus I_{m_0}^>}(N^>) \Omega_{\text{bg}}),$$

where

$$I_{m_0}^> = [n_{\text{bg}}^>(L)L^d - m_0 L^d, n_{\text{bg}}^>(L)L^d + m_0 L^d].$$

Lemma 5.3 therefore gives the claimed bound.

b) By the Cauchy-Schwarz inequality,

$$\begin{aligned} n_{0,\text{bg}}(L) &\geq -L^{-d} (\Omega_{\text{bg}} | \mathbb{1}_{]-\infty, 0[}(N_{\mathbf{0}}^{\text{ext}}) | N_{\mathbf{0}}^{\text{ext}} | \Omega_{\text{bg}}) \\ &\geq -\sqrt{(\Omega_{\text{bg}} | \mathbb{1}_{]-\infty, 0[}(N_{\mathbf{0}}^{\text{ext}}) \Omega_{\text{bg}})} L^{-d} \sqrt{(\Omega_{\text{bg}} | (N_{\mathbf{0}}^{\text{ext}})^2 \Omega_{\text{bg}})}. \end{aligned}$$

The pair distribution of the Bogoliubov ground state gives

$$\begin{aligned} L^{-2d}(\Omega_{\text{bg}}|(N_{\mathbf{0}}^{\text{ext}})^2\Omega_{\text{bg}}) &= n_{0,\text{bg}}(L)^2 + 2L^{-2d} \sum_{\mathbf{p} \in \Xi_L^{\geq}} s_{\mathbf{p}}^2(1 + s_{\mathbf{p}}^2) \\ &= n_{0,\text{bg}}(L)^2 + \begin{cases} O(L^{-2} \log L), & d = 2, \\ O(L^{-3}), & d = 3. \end{cases} \end{aligned}$$

Here we used Lemma 5.1 and $\sum_{\mathbf{p} \in \Xi_L^{\geq}} s_{\mathbf{p}}^2 = L^d n_{\text{bg}}^{\geq}(L) = O(L^d)$. Thus

$$L^{-d} \sqrt{(\Omega_{\text{bg}}|(N_{\mathbf{0}}^{\text{ext}})^2\Omega_{\text{bg}})} \longrightarrow |n_{0,\text{bg}}|.$$

Since $n_{0,\text{bg}}(L) \rightarrow n_{0,\text{bg}} < 0$, we have $n_{0,\text{bg}}(L) < 0$ for all sufficiently large L . Rearranging the Cauchy-Schwarz bound yields

$$\sqrt{(\Omega_{\text{bg}}|\mathbb{1}_{]-\infty,0[}(N_{\mathbf{0}}^{\text{ext}})\Omega_{\text{bg}})} \geq \frac{-n_{0,\text{bg}}(L)}{L^{-d} \sqrt{(\Omega_{\text{bg}}|(N_{\mathbf{0}}^{\text{ext}})^2\Omega_{\text{bg}})}} \longrightarrow 1.$$

As the left-hand side is at most 1, this proves b). $\square$

Part a) shows that, for every fixed $m_0 > 0$, the Bogoliubov ground state is concentrated in the spectral subspace

$$\mathbb{1}_{I_{m_0}}(N_{\mathbf{0}}^{\text{ext}})\mathcal{H}^{\text{ext}}.$$

This justifies interpreting $n_{0,\text{bg}}$ as the condensate density: when $n_{0,\text{bg}} > 0$, the zero-mode occupation is concentrated at macroscopically large values. Choosing $0 < m_0 < n_{0,\text{bg}}$ ensures that $I_{m_0} \subset]0, +\infty[$ for all sufficiently large L . Consequently, the unphysical tail $\mathbb{1}_{]-\infty,0[}(N_{\mathbf{0}}^{\text{ext}})\Omega_{\text{bg}}$ is exponentially suppressed in L^{d-1} .

Remark 5.5. It is clear from the proof of Lemma 5.3 and Corollary 5.4 that one gets similar exponential decay even for vanishing condensate densities

$$\lim_{L \rightarrow +\infty} n_{\text{bg},0}(L) \quad (5.18)$$

and for spectral intervals of the form

$$I_{m_0,\epsilon} = [n_{0,\text{bg}}(L)L^d - m_0L^{d-1+\epsilon}, n_{0,\text{bg}}(L)L^d + m_0L^{d-1+\epsilon}], \quad \epsilon > 0. \quad (5.19)$$

A complementary lower bound shows that, for $d = 2, 3$, the power L^{d-1} in the exponential cannot be improved.

Proposition 5.6. *Fix $d \geq 1$ and assume Assumption 2.1. Assume that the Bogoliubov vacuum Ω_{bg} is well defined. Then, for every $\lambda > 1$ and all sufficiently large N ,*

$$(\Omega_{\text{bg}}|\mathbb{1}_{]-\infty,0[}(N_{\mathbf{0}}^{\text{ext}})\Omega_{\text{bg}}) \geq \exp \left\{ -\lambda n \frac{2\pi}{\sqrt{\nu}} L^{d-1} \right\}.$$

Proof. Let $(\mathbf{k}_1)_i = \frac{2\pi}{L} \delta_{i,1}$. The marginal distribution of $N_{\mathbf{k}_1}$ is geometric. Thus

$$(\Omega_{\text{bg}}|\mathbb{1}_{]-\infty,0[}(N_{\mathbf{0}}^{\text{ext}})\Omega_{\text{bg}}) \geq (\Omega_{\text{bg}}|\mathbb{1}_{[\lfloor \frac{N}{2} \rfloor + 1, +\infty[}(N_{\mathbf{k}_1})\Omega_{\text{bg}}) = (\tanh^2 \theta_{\mathbf{k}_1})^{\lfloor \frac{N}{2} \rfloor + 1}.$$

As $L \rightarrow +\infty$, $\mathbf{k}_1 \rightarrow 0$. Therefore, we can expand

$$\tanh^2 \theta_{\mathbf{k}_1} = 1 - \frac{4\pi}{\sqrt{\nu}L} + o(L^{-1}).$$

Consequently,

$$(\lfloor \frac{N}{2} \rfloor + 1) \log(\tanh^2 \theta_{\mathbf{k}_1}) = -\frac{2\pi}{\sqrt{\nu}} n L^{d-1} (1 + o(1))$$

For each fixed $\lambda > 1$, the claimed lower bound follows for all sufficiently large N . $\square$

For $d = 1$, this proves

$$\liminf_{N \rightarrow \infty} (\Omega_{\text{bg}} | \mathbb{1}_{[-\infty, 0[} (N_{\mathbf{0}}^{\text{ext}}) \Omega_{\text{bg}}) \geq e^{-\frac{2\pi n}{\sqrt{\nu}}} > 0.$$

Thus the unphysical tail does not tend to zero in dimension one.

The estimate in Corollary 5.4 remains valid after applying a fixed number of creation and annihilation operators and inserting a polynomial weight in the extended zero-mode occupation number.

Proposition 5.7. *Fix $d = 2, 3$ and suppose that Assumptions 2.1 and 2.3 hold. Let $m_0 > 0$ and $\alpha \geq 0$ be fixed, let $m_0 \geq 0$ be a fixed integer, and set*

$$I_{m_0} = [n_{0,\text{bg}} L^d - m_0 L^d, n_{0,\text{bg}} L^d + m_0 L^d]. \quad (5.20)$$

Then, for all $\delta \in]0, 1[$ and, for all sufficiently large L ,

$$\begin{aligned} (a_{\mathbf{k}_1}^? \cdots a_{\mathbf{k}_m}^? \Omega_{\text{bg}} | | N_{\mathbf{0}}^{\text{ext}} |^\alpha \mathbb{1}_{\mathbb{R} \setminus I_{m_0}} (N_{\mathbf{0}}^{\text{ext}}) a_{\mathbf{k}_1}^? \cdots a_{\mathbf{k}_m}^? \Omega_{\text{bg}}) \\ \leq \exp \left\{ -\frac{\pi(1+\delta)m_0}{\sqrt{\nu}} L^{d-1} \right\}. \end{aligned} \quad (5.21)$$

The estimate is uniform in $\mathbf{k}_1, \dots, \mathbf{k}_m \in \Xi_L^\geq$, including repeated momenta, and in the choices $a_{\mathbf{k}}^? \in \{a_{\mathbf{k}}, a_{\mathbf{k}}^*\}$.

Proof. Let s be the number of creation operators minus the number of annihilation operators, so that $|s| \leq m$. Since $N_{\mathbf{0}}^{\text{ext}} = N - N^>$, spectral calculus gives

$$|N_{\mathbf{0}}^{\text{ext}}|^\alpha \mathbb{1}_{\mathbb{R} \setminus I_{m_0}} (N_{\mathbf{0}}^{\text{ext}}) a_{\mathbf{k}_1}^? \cdots a_{\mathbf{k}_m}^? = a_{\mathbf{k}_1}^? \cdots a_{\mathbf{k}_m}^? |N_{\mathbf{0}}^{\text{ext}} - s|^\alpha \mathbb{1}_{(\mathbb{R} \setminus I_{m_0})+s} (N_{\mathbf{0}}^{\text{ext}}).$$

Successively applying the usual creation and annihilation bounds on each eigenspace of $N^>$ yields

$$0 \leq (a_{\mathbf{k}_m}^?)^* \cdots (a_{\mathbf{k}_1}^?)^* a_{\mathbf{k}_1}^? \cdots a_{\mathbf{k}_m}^? \leq (N^> + m + 1)^m.$$

This product commutes with $N^>$ and $N_{\mathbf{0}}^{\text{ext}}$.

We first bound the polynomial moments needed below. Equation (2.80) implies, for sufficiently large L ,

$$\log(\Omega_{\text{bg}} | e^{\frac{N^>}{L^d}} \Omega_{\text{bg}}) = -\frac{1}{2} \sum_{\mathbf{p} \in \Xi_L^\geq} \log(1 - (e^{\frac{2}{L^d}} - 1) s_{\mathbf{p}}^2) \leq CL^{-d} \sum_{\mathbf{p} \in \Xi_L^\geq} s_{\mathbf{p}}^2 \leq C.$$

Since $N = O(L^d)$ and for any $r \geq 0$

$$\left(\frac{N^>}{L^d} \right)^r \leq D_r e^{\frac{N^>}{L^d}} \quad (5.22)$$

For some positive constant $D_r > 0$ independent from L , we obtain

$$(\Omega_{\text{bg}} | (N + N^> + m + 1)^r \Omega_{\text{bg}}) \leq C_r L^{dr}, \quad r \geq 0,$$

where $C_r > 0$ is again independent from the volume. In particular, using the preceding operator bound and then Cauchy-Schwarz, the matrix element in the statement is at most

$$\begin{aligned} & \left\| |N_{\mathbf{0}}^{\text{ext}} - s|^\alpha (a_{\mathbf{k}_m}^?)^* \cdots (a_{\mathbf{k}_1}^?)^* a_{\mathbf{k}_1}^? \cdots a_{\mathbf{k}_m}^? \Omega_{\text{bg}} \right\| \left\| \mathbb{1}_{(\mathbb{R} \setminus I_{m_0})+s} (N_{\mathbf{0}}^{\text{ext}}) \Omega_{\text{bg}} \right\| \\ & \leq (\Omega_{\text{bg}} | (N + N^> + m + 1)^{2(m+\alpha)} \Omega_{\text{bg}})^{\frac{1}{2}} \left\| \mathbb{1}_{(\mathbb{R} \setminus I_{m_0})+s} (N_{\mathbf{0}}^{\text{ext}}) \Omega_{\text{bg}} \right\| \\ & \leq CL^{d(m+\alpha)} \left\| \mathbb{1}_{(\mathbb{R} \setminus I_{m_0})+s} (N_{\mathbf{0}}^{\text{ext}}) \Omega_{\text{bg}} \right\|. \end{aligned}$$

The polynomial factor in L can be absorbed into the stated exponential rates by applying Corollary 5.4 with a smaller $\delta \in (0, 1)$. This proves the claim. $\square$

5.2. Estimates for the tail remainders. Throughout this subsection, the external momenta are bounded away from zero, and limits are taken along admissible lattice momenta. In addition to the preceding assumptions, we assume the uniform summability bound

$$\limsup_{L \rightarrow \infty} \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} |\hat{v}(\mathbf{p})| (|c_{\mathbf{p}} s_{\mathbf{p}}| + s_{\mathbf{p}}^2) < \infty.$$

It holds for $d = 2, 3$ under the additional Assumption 3.1 by standard Sobolev inequalities.

The following lemma shows that the Hellmann–Feynman contribution of $R_{4,2}^L$ vanishes in the thermodynamic limit. We retain the operator ordering in its definition in Section 2.6.

Lemma 5.8. *Fix $d = 2, 3$ and assume Assumptions 2.1 and 2.3, together with the uniform summability bound stated above. Let*

$$R_{4,2}^L = \frac{1}{2L^d} \left(\sqrt{N_{\mathbf{0}}^{\text{ext}}(N_{\mathbf{0}}^{\text{ext}} - 1)} - N_{\mathbf{0}}^{\text{ext}} + \frac{1}{2} \right) \sum_{\mathbf{p} \in \Xi_L^>} \hat{v}(\mathbf{p}) A_{\mathbf{p}}^* + \text{h.c.}$$

Then

$$\lim_{L \rightarrow \infty} \left[(b_{\mathbf{k}}^* \Omega_{\text{bg}} | R_{4,2}^L b_{\mathbf{k}}^* \Omega_{\text{bg}}) - (\Omega_{\text{bg}} | R_{4,2}^L \Omega_{\text{bg}}) \right] = 0.$$

Proof. Let

$$f(x) := \sqrt{|x(x-1)|} - x + \frac{1}{2}.$$

As $x \rightarrow +\infty$, this function satisfies $f(x) = O(x^{-1})$, $f(x+1) - f(x) = O(x^{-2})$, $f(x+1) - 2f(x) + f(x-1) = O(x^{-3})$. Since $[N_{\mathbf{0}}^{\text{ext}}, A_{\mathbf{p}}] = 2A_{\mathbf{p}}$, we have $A_{\mathbf{p}} f(N_{\mathbf{0}}^{\text{ext}}) = f(N_{\mathbf{0}}^{\text{ext}} - 2) A_{\mathbf{p}}$. Consequently, the double-commutator identity gives

$$(b_{\mathbf{k}}^* \Omega_{\text{bg}} | R_{4,2}^L b_{\mathbf{k}}^* \Omega_{\text{bg}}) - (\Omega_{\text{bg}} | R_{4,2}^L \Omega_{\text{bg}}) = \frac{\kappa}{L^d} \text{Re} \sum_{\mathbf{p} \in \Xi_L^>} \hat{v}(\mathbf{p}) (\Omega_{\text{bg}} | [b_{\mathbf{k}}, [f(N_{\mathbf{0}}^{\text{ext}} - 2) A_{\mathbf{p}}, b_{\mathbf{k}}^*]] \Omega_{\text{bg}}).$$

We expand the commutators as follows:

$$\begin{aligned} [b_{\mathbf{k}}, [f(N_{\mathbf{0}}^{\text{ext}} - 2) A_{\mathbf{p}}, b_{\mathbf{k}}^*]] &= f(N_{\mathbf{0}}^{\text{ext}} - 2) [b_{\mathbf{k}}, [A_{\mathbf{p}}, b_{\mathbf{k}}^*]] + [b_{\mathbf{k}}, f(N_{\mathbf{0}}^{\text{ext}} - 2)] [A_{\mathbf{p}}, b_{\mathbf{k}}^*] \\ &\quad + [f(N_{\mathbf{0}}^{\text{ext}} - 2), b_{\mathbf{k}}^*] [b_{\mathbf{k}}, A_{\mathbf{p}}] + [b_{\mathbf{k}}, [f(N_{\mathbf{0}}^{\text{ext}} - 2), b_{\mathbf{k}}^*]] A_{\mathbf{p}}. \end{aligned}$$

Using the expansion of $A_{\mathbf{p}}$ in $M_{\mathbf{p}}$ and $B_{\mathbf{p}}$, together with (2.72) and (2.74), we obtain the four identities

$$\begin{aligned} &(\Omega_{\text{bg}} | f(N_{\mathbf{0}}^{\text{ext}} - 2) [b_{\mathbf{k}}, [A_{\mathbf{p}}, b_{\mathbf{k}}^*]] \Omega_{\text{bg}}) \\ &= -s_{\mathbf{k}} c_{\mathbf{k}} (\delta_{\mathbf{p}, \mathbf{k}} + \delta_{-\mathbf{p}, \mathbf{k}}) (\Omega_{\text{bg}} | f(N_{\mathbf{0}}^{\text{ext}} - 2) \Omega_{\text{bg}}), \end{aligned} \tag{5.23}$$

$$\begin{aligned} &(\Omega_{\text{bg}} | [b_{\mathbf{k}}, f(N_{\mathbf{0}}^{\text{ext}} - 2)] [A_{\mathbf{p}}, b_{\mathbf{k}}^*] \Omega_{\text{bg}}) \\ &= -s_{\mathbf{k}} c_{\mathbf{k}} (\delta_{\mathbf{p}, \mathbf{k}} + \delta_{-\mathbf{p}, \mathbf{k}}) (\Omega_{\text{bg}} | [b_{\mathbf{k}}, f(N_{\mathbf{0}}^{\text{ext}} - 2)] b_{\mathbf{k}}^* \Omega_{\text{bg}}), \end{aligned} \tag{5.24}$$

$$\begin{aligned} &(\Omega_{\text{bg}} | [f(N_{\mathbf{0}}^{\text{ext}} - 2), b_{\mathbf{k}}^*] [b_{\mathbf{k}}, A_{\mathbf{p}}] \Omega_{\text{bg}}) \\ &= s_{\mathbf{k}}^2 (\delta_{\mathbf{p}, \mathbf{k}} + \delta_{-\mathbf{p}, \mathbf{k}}) (\Omega_{\text{bg}} | [f(N_{\mathbf{0}}^{\text{ext}} - 2), b_{\mathbf{k}}^*] b_{-\mathbf{k}}^* \Omega_{\text{bg}}), \\ &(\Omega_{\text{bg}} | [b_{\mathbf{k}}, [f(N_{\mathbf{0}}^{\text{ext}} - 2), b_{\mathbf{k}}^*]] A_{\mathbf{p}} \Omega_{\text{bg}}) \end{aligned} \tag{5.25}$$

$$= s_{\mathbf{p}}^2 (\Omega_{\text{bg}} | [b_{\mathbf{k}}, [f(N_{\mathbf{0}}^{\text{ext}} - 2), b_{\mathbf{k}}^*]] B_{\mathbf{p}}^* \Omega_{\text{bg}}) - s_{\mathbf{p}} c_{\mathbf{p}} (\Omega_{\text{bg}} | [b_{\mathbf{k}}, [f(N_{\mathbf{0}}^{\text{ext}} - 2), b_{\mathbf{k}}^*]] \Omega_{\text{bg}}). \tag{5.26}$$

Now, we estimate the scalar product in Equation 5.23. Choose $0 < m_0 < n_{0, \text{bg}}$ and set

$$I_{m_0} := [n_{0, \text{bg}} L^d - m_0 L^d, n_{0, \text{bg}} L^d + m_0 L^d].$$

Using Proposition 5.7, the first expectation in (5.23) satisfies

$$\begin{aligned} |(\Omega_{\text{bg}}|f(N_{\mathbf{0}}^{\text{ext}} - 2)\Omega_{\text{bg}})| &\leq |(\Omega_{\text{bg}}|\mathbb{1}_{I_{m_0}}(N_{\mathbf{0}}^{\text{ext}})f(N_{\mathbf{0}}^{\text{ext}} - 2)\Omega_{\text{bg}})| \\ &+ |(\Omega_{\text{bg}}|\mathbb{1}_{\mathbb{R} \setminus I_{m_0}}(N_{\mathbf{0}}^{\text{ext}})f(N_{\mathbf{0}}^{\text{ext}} - 2)\Omega_{\text{bg}})| = O(L^{-d}) + O(e^{-cL^{d-1}}). \end{aligned} \quad (5.27)$$

Thus

$$\frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} \hat{v}(\mathbf{p})(\Omega_{\text{bg}}|f(N_{\mathbf{0}}^{\text{ext}} - 2)[b_{\mathbf{k}}, [A_{\mathbf{p}}, b_{\mathbf{k}}^*]]\Omega_{\text{bg}}) = O(L^{-2d}).$$

Similarly, it follows that

$$\begin{aligned} |(\Omega_{\text{bg}}|[b_{\mathbf{k}}, [f(N_{\mathbf{0}}^{\text{ext}} - 2), b_{\mathbf{k}}^*]]\Omega_{\text{bg}})| &= O(L^{-2d}), \\ |(\Omega_{\text{bg}}|[f(N_{\mathbf{0}}^{\text{ext}} - 2), b_{\mathbf{k}}^*]b_{-\mathbf{k}}^*\Omega_{\text{bg}})| &= O(L^{-2d}), \\ |(\Omega_{\text{bg}}|[b_{\mathbf{k}}, f(N_{\mathbf{0}}^{\text{ext}} - 2)]b_{\mathbf{k}}^*\Omega_{\text{bg}})| &= O(L^{-2d}). \end{aligned}$$

The remaining sum over momenta is controlled by the Kronecker deltas for the first three terms, and by the uniform summability assumption for the fourth one. Consequently,

$$\left| \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} \hat{v}(\mathbf{p})(\Omega_{\text{bg}}|[b_{\mathbf{k}}, [f(N_{\mathbf{0}}^{\text{ext}} - 2)A_{\mathbf{p}}, b_{\mathbf{k}}^*]]\Omega_{\text{bg}}) \right| = O(L^{-2d}),$$

which proves the lemma. $\square$

An analogous proof shows

Lemma 5.9. *Under the assumptions of Lemma 5.8, let*

$$R_{4,1}^L = \frac{1}{2L^d} \left(\sqrt{N_{\mathbf{0}}^+(N_{\mathbf{0}}^+ - 1)} - \sqrt{N_{\mathbf{0}}^{\text{ext}}(N_{\mathbf{0}}^{\text{ext}} - 1)} \right) \sum_{\mathbf{p} \in \Xi_L^>} \hat{v}(\mathbf{p})A_{\mathbf{p}}^* + \text{h.c.}$$

Then

$$\lim_{L \rightarrow \infty} \left[(b_{\mathbf{k}}^*\Omega_{\text{bg}}|R_{4,1}^L b_{\mathbf{k}}^*\Omega_{\text{bg}}) - (\Omega_{\text{bg}}|R_{4,1}^L \Omega_{\text{bg}}) \right] = 0.$$

The same arguments apply to normalized states $\mathcal{N}b_{\mathbf{k}_1}^* \cdots b_{\mathbf{k}_n}^* \Omega_{\text{bg}}$ with a fixed number of quasiparticles and momenta bounded away from zero. Commuting the finitely many creation and annihilation operators introduces only finitely many integer shifts and finitely excited vectors, to which the same tail estimates apply. Thus the conclusions of Lemmas 5.8 and 5.9, as well as those of Lemmas 2.5, 2.6 and 2.7, extend to these states. In particular, neither $R_{4,1}^L$ nor $R_{4,2}^L$ contributes to the Hellmann–Feynman correction in the thermodynamic limit.

6. ESTIMATES ON PERTURBATIVE CORRECTIONS

In this section we give rigorous proofs of the main results of our paper presented in Section 3. First we analyze rigorously the Hellmann–Feynman contribution to the energy shift for low momenta. In the following subsections we will make rigorous the arguments leading to Proposition 3.4 about the Fermi golden rule correction, and thus, to Theorem 3.5 about the dispersion relation shift in the canonical setting. Several auxiliary functions are introduced in Subsections 6.2 and 6.3. In Subsection 6.4 we partition the integration domain into three regions adapted to the singularity structure of the integrals. The corresponding contributions are estimated in Subsection 6.5.

6.1. Estimates for the Hellmann-Feynman term.

Remark 6.1. (ν -dependence) We study the small-momentum asymptotics of $\delta_4^{\text{can}}(\mathbf{k}, \nu, \hat{v})$ and $\delta_3^{\text{can}}(\mathbf{k}, \nu, \hat{v})$ in the regime $\frac{|\mathbf{k}|}{\sqrt{\nu}} \rightarrow 0$. Since we will eventually consider the dilute scaling $\nu \mapsto \kappa\nu$ and the limit $\kappa\nu \rightarrow 0$, see Sec. 3.5, we must keep track of the dependence on ν .

After the rescaling $p \mapsto \sqrt{\nu} p$, the Bogoliubov dispersion relation becomes

$$\omega_{\text{bg}}(\sqrt{\nu} p) = \nu \sqrt{\frac{p^4}{4} + \frac{\hat{v}(\sqrt{\nu} p)}{\hat{v}(0)} p^2}. \quad (6.1)$$

Thus, apart from explicit powers of ν , the remaining dependence on ν enters through $\hat{v}(\sqrt{\nu} \cdot)$. Notice also that each derivative of the potential with respect to the rescaled variable produces an additional factor of $\sqrt{\nu}$.

Accordingly, we shall perform the computation with $\hat{v}(0) = 1$ and $\nu = 1$. The correct dependence is restored by multiplying by the global factor $\hat{v}(0)\nu^{3/2}$ and making the replacements

$$k \mapsto \frac{k}{\sqrt{\nu}}$$

and

$$\hat{v}(\cdot) \mapsto \frac{\hat{v}(\sqrt{\nu} \cdot)}{\hat{v}(0)}.$$

Before the proof of Proposition 3.2, we need the following simple lemma:

Lemma 6.2. *Let $d = 3$ and f be a radial function in $L^2(\mathbb{R}^3)$. Then*

$$\frac{1}{(2\pi)^3} \int d\mathbf{p} f(\mathbf{p}) \left(\hat{v}(\mathbf{p}) - \frac{\hat{v}(\mathbf{p} + \mathbf{k}) + \hat{v}(\mathbf{p} - \mathbf{k})}{2} \right) \quad (6.2)$$

$$= -\frac{k^2}{12\pi^2} \int_0^\infty dp f(p) (\hat{v}''(p)p^2 + 2\hat{v}'(p)p) + o(k^2). \quad (6.3)$$

Proof.

$$\begin{aligned} -(6.2) &= \frac{1}{2} \sum_{j,i=1}^3 k_j k_i \int_{-1}^1 dv (1 - |v|) \frac{1}{(2\pi)^3} \int d\mathbf{p} f(\mathbf{p}) (\partial_j \partial_i \hat{v})(\mathbf{p} + v\mathbf{k}) \\ &= \frac{1}{2} \sum_{j,i=1}^3 k_j k_i \frac{1}{(2\pi)^3} \int d\mathbf{p} f(\mathbf{p}) (\partial_j \partial_i \hat{v})(\mathbf{p}) + o(k^2). \end{aligned} \quad (6.4)$$

Equation (6.4) can be justified by an L^2 -translation continuity argument. We estimate

$$\begin{aligned} &\left| \int d\mathbf{p} f(\mathbf{p}) (\partial_j \partial_i \hat{v})(\mathbf{p}) - \int d\mathbf{p} f(\mathbf{p}) \int_{-1}^1 dv (1 - |v|) (\partial_j \partial_i \hat{v})(\mathbf{p} + v\mathbf{k}) \right| \\ &\leq \|f\|_{L^2(\mathbb{R}^3)} \sup_{|v| \leq 1} \|\partial_i \partial_j \hat{v}(\cdot + v\mathbf{k}) - \partial_i \partial_j \hat{v}\|_{L^2(\mathbb{R}^3)} \end{aligned} \quad (6.5)$$

Now, for every $g \in L^2(\mathbb{R}^3)$ and $\varphi \in C_c^\infty(\mathbb{R}^3)$,

$$\sup_{|\mathbf{q}| \leq \mathbf{k}} \|g(\cdot + \mathbf{q}) - g\|_{L^2} \leq 2\|g - \varphi\|_{L^2} + k\|\nabla \varphi\|_{L^2}. \quad (6.6)$$

By density, the left-hand side therefore vanishes as $k \rightarrow 0$. Since $|v\mathbf{k}| \leq |\mathbf{k}|$, the right-hand side of (6.5) is $o(1)$, which yields the $o(k^2)$ remainder in (6.4).

Then,

$$\sum_{i,j=1}^3 \frac{k_i k_j}{k^2} \partial_i \partial_j \hat{v}(\mathbf{p}) = \hat{v}''(p) \cos^2 \theta + \frac{\hat{v}'(p)}{p} (1 - \cos^2 \theta), \quad (6.7)$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{k}$. Thus the main term of (6.4) is

$$\begin{aligned} & -\frac{k^2}{2(2\pi)^3} \int_0^\infty dp f(p) p^2 \int_{-1}^1 d(\cos \theta) \int_0^{2\pi} d\varphi \left(\hat{v}''(p) \cos^2 \theta + \frac{\hat{v}'(p)}{p} (1 - \cos^2 \theta) \right) \\ & = (6.3). \end{aligned} \tag{6.8}$$

□

Proof of Proposition 3.2. Let us fix without loss of generality $\hat{v}(0) = 1$. On account of Remark 6.1 we will also take $\nu = 1$. We rewrite the Hellmann-Feynman contribution (3.15) in the thermodynamic limit as in Equation (3.16)

$$\begin{aligned} & \lim_{L \rightarrow +\infty} (\Omega_{\text{bg}} | [b_{\mathbf{k}}, [H_{4,2}^L, b_{\mathbf{k}}^*]] \Omega_{\text{bg}}) = \\ & = (c_{\mathbf{k}} + s_{\mathbf{k}})^2 \frac{1}{(2\pi)^3} \int d\mathbf{p} c_{\mathbf{p}} s_{\mathbf{p}} \hat{v}(\mathbf{p}) \end{aligned} \tag{6.9}$$

$$+ (c_{\mathbf{k}}^2 + s_{\mathbf{k}}^2) \frac{1}{(2\pi)^3} \int d\mathbf{p} s_{\mathbf{p}}^2 \left(-\hat{v}(\mathbf{p}) + \frac{\hat{v}(\mathbf{p} + \mathbf{k}) + \hat{v}(\mathbf{p} - \mathbf{k})}{2} \right) \tag{6.10}$$

$$+ 2c_{\mathbf{k}} s_{\mathbf{k}} \frac{1}{(2\pi)^3} \int d\mathbf{p} c_{\mathbf{p}} s_{\mathbf{p}} \left(-\hat{v}(\mathbf{p}) + \frac{\hat{v}(\mathbf{p} + \mathbf{k}) + \hat{v}(\mathbf{p} - \mathbf{k})}{2} \right) \tag{6.11}$$

$$- (c_{\mathbf{k}} - s_{\mathbf{k}})^2 \hat{v}(\mathbf{k}) \frac{1}{(2\pi)^3} \int d\mathbf{p} s_{\mathbf{p}}^2. \tag{6.12}$$

Note that the convergence of the Riemann sums is implied by

$$\hat{v}(\mathbf{p}) \in C^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3), \quad \partial_i \hat{v}(\mathbf{p}) \in L^2(\mathbb{R}^3) \tag{6.13}$$

and

$$|c_{\mathbf{p}} s_{\mathbf{p}}| = O\left(\frac{|\hat{v}(\mathbf{p})|}{|\mathbf{p}|^2}\right), \quad s_{\mathbf{p}}^2 = O\left(\frac{\hat{v}(\mathbf{p})^2}{|\mathbf{p}|^4}\right), \text{ as } |\mathbf{p}| \rightarrow +\infty. \tag{6.14}$$

The first line (6.9) is the infrared divergent term. The second and third lines, (6.10) and (6.11), can be treated as in Lemma 6.2. They give

$$(6.10) = (c_{\mathbf{k}}^2 + s_{\mathbf{k}}^2) \frac{k^2}{12\pi^2} \int dp s_p^2 (\hat{v}''(p) p^2 + 2p \hat{v}'(p)) + o(k), \tag{6.15}$$

$$(6.11) = c_{\mathbf{k}} s_{\mathbf{k}} \frac{k^2}{6\pi^2} \int dp c_p s_p (\hat{v}''(p) p^2 + 2p \hat{v}'(p)) + o(k). \tag{6.16}$$

To conclude the proof we just have to re-organize the coefficients $c_{\mathbf{k}} s_{\mathbf{k}}$, $(c_{\mathbf{k}} \pm s_{\mathbf{k}})$ and $c_{\mathbf{k}}^2 + s_{\mathbf{k}}^2$ to recover the expression for c_4 and for the infrared term. Using Equation (3.22) we have

$$c_{\mathbf{k}} s_{\mathbf{k}} = \frac{\sqrt{\nu}}{2k} (1 + O(k^2)) \tag{6.17}$$

$$(c_{\mathbf{k}} - s_{\mathbf{k}})^2 = \frac{k}{2\sqrt{\nu}} (1 + O(k^2)) \tag{6.18}$$

$$(c_{\mathbf{k}} + s_{\mathbf{k}})^2 = \frac{2\omega_{\text{bg}}(k)}{k^2} \tag{6.19}$$

$$c_{\mathbf{k}}^2 + s_{\mathbf{k}}^2 = \frac{\sqrt{\nu}}{k} (1 + O(k)) \tag{6.20}$$

After restoring the correct powers of ν and $\hat{v}(0)$, we conclude the proof. □

6.2. Bogoliubov coefficients in momentum coordinates. In this subsection, we describe several useful properties and expansions of the Bogoliubov coefficients $c_{\mathbf{k}}$, $s_{\mathbf{k}}$, and related functions. The proofs of the results in this subsection are simple but tedious, therefore we will omit them.

Regularized Bogoliubov coefficients. In view of Remark 6.1, we express all momenta in units of $\sqrt{\nu}$. Accordingly, throughout this section we carry out the computation under the normalization $\nu = 1$ and $\hat{v}(0) = 1$. Equivalently, we perform the replacements $\hat{v}(p) \equiv \frac{\hat{v}(\sqrt{\nu}p)}{\hat{v}(0)}$, $\omega_{\text{bg}}(p) \equiv \frac{\omega_{\text{bg}}(\sqrt{\nu}p)}{\nu}$, $s_{\mathbf{p}} \equiv s_{\sqrt{\nu}\mathbf{p}}$ and $c_{\mathbf{p}} \equiv c_{\sqrt{\nu}\mathbf{p}}$.

The coefficients $c_{\mathbf{p}}$ and $s_{\mathbf{p}}$ diverge as $\frac{1}{\sqrt{2\omega_{\text{bg}}(p)}}$ as $p \rightarrow 0$. Thus, it will be more convenient to work with regularized versions of the latter coefficients in the following

$$\begin{aligned} c(p) &:= \sqrt{2\omega_{\text{bg}}(p)} c_{\mathbf{p}} = \sqrt{\sqrt{\omega_{\text{bg}}(p)^2 + \hat{v}(p)^2} + \omega_{\text{bg}}(p)} \\ &= \sqrt{\frac{p^2}{2} + \hat{v}(p) + \omega_{\text{bg}}(p)}, \end{aligned} \quad (6.21)$$

$$\begin{aligned} s(p) &:= \sqrt{2\omega_{\text{bg}}(p)} s_{\mathbf{p}} = \text{sign}(\hat{v}(p)) \sqrt{\sqrt{\omega_{\text{bg}}(p)^2 + \hat{v}(p)^2} - \omega_{\text{bg}}(p)} \\ &= \text{sign}(\hat{v}(p)) \sqrt{\frac{p^2}{2} + \hat{v}(p) - \omega_{\text{bg}}(p)}. \end{aligned} \quad (6.22)$$

Unlike c_p and s_p , the regularized coefficients $c(p)$ and $s(p)$ extend to C^5 functions in a neighborhood of the origin. In particular, $c(0) = s(0) = 1$. We collect their main properties in the following lemma.

Lemma 6.3. *Suppose that the potential satisfies Assumptions 2.1 and 3.1.*

- a) *The regularized Bogoliubov coefficients satisfy the following relations for all $p \in [0, +\infty[$*

$$c(p)^2 - s(p)^2 = 2\omega_{\text{bg}}(p) \quad (6.23)$$

$$c(p)^2 + s(p)^2 = p^2 + 2\hat{v}(p) \quad (6.24)$$

$$c(p)s(p) = \hat{v}(p) \quad (6.25)$$

$$c(p) - s(p) = p \quad (6.26)$$

$$c(p) + s(p) = \frac{2\omega_{\text{bg}}(p)}{p} \quad (6.27)$$

- b) *The regularized Bogoliubov coefficient have the following expansions*

$$\hat{v}(p) = 1 + r(p), \quad r(p) \in O(p^2) \quad (6.28)$$

$$c(p) = 1 + \frac{\omega_{\text{bg}}(p)}{2} + \frac{\omega_{\text{bg}}(p)^2}{8} + \frac{r(p)}{2} + R_c(p), \quad (6.29)$$

$$s(p) = 1 - \frac{\omega_{\text{bg}}(p)}{2} + \frac{\omega_{\text{bg}}(p)^2}{8} + \frac{r(p)}{2} + R_s(p), \quad (6.30)$$

as $p \rightarrow 0$, where R_c and R_s are C^2 and of order $O(p^3)$ in neighborhood of 0.

We next record the large-momentum behavior of $c(p)$, $s(p)$, and their derivatives.

Lemma 6.4. *Suppose that the potential satisfies Assumptions 2.1 and 3.1. Then, we have the following*

a) The rescaled Bogoliubov energy and its first and second derivatives satisfy the following

$$\omega_{\text{bg}}(p) = \frac{p^2}{2} \left(1 + O\left(\frac{1}{p^2}\right) \right), \quad (6.31)$$

$$\omega'_{\text{bg}}(p) = p \left(1 + O\left(\frac{1}{p}\right) \right), \quad (6.32)$$

$$\omega''_{\text{bg}}(p) = O(1) \quad (6.33)$$

as $p \rightarrow +\infty$.

b) The regularized coefficients $s(p)$, $c(p)$ and their derivatives are continuous functions satisfying

$$c(p) = p \left(1 + O\left(\frac{1}{p^2}\right) \right), \quad s(p) = \frac{\hat{v}(p)}{p} \left(1 + O\left(\frac{1}{p^2}\right) \right) \quad (6.34)$$

$$c'(p) = 1 + O\left(\frac{1}{p}\right), \quad s'(p) = O\left(\frac{|\hat{v}'(p)|}{p} + \frac{|\hat{v}(p)|}{p^2}\right) \quad (6.35)$$

$$c''(p) = O\left(\frac{1}{p}\right), \quad s''(p) = O\left(\frac{|\hat{v}''(p)|p^2 + |\hat{v}'(p)|p + |\hat{v}(p)|}{p^3}\right) \quad (6.36)$$

as $p \rightarrow +\infty$.

The symmetric combination of two translated Bogoliubov energies admits the following Taylor expansion with an exact remainder:

$$\omega_{\text{bg}}(p + \epsilon) + \omega_{\text{bg}}(p - \epsilon) = 2\omega_{\text{bg}}(p) + \epsilon^2 R_\omega(p, \epsilon), \quad (6.37)$$

$$R_\omega(p, \epsilon) = \int_{-1}^1 dv (1 - |v|) \omega''_{\text{bg}}(p + v\epsilon), \quad (6.38)$$

for $p \geq \epsilon$. Clearly, $R_\omega(p, 0) = \omega''_{\text{bg}}(p)$, for $p > 0$.

Lemma 6.4 immediately yields the following Corollary

Corollary 6.5. Suppose that the potential satisfies Assumptions 2.1 and 3.1. Then

$$|R_\omega(p, \epsilon)| \leq C, \quad (6.39)$$

for $p \geq \epsilon > 0$ and for some sufficiently large constant $C > 0$.

Regularized Fermi golden rule integrands. We proceed to define regularized versions of $j(k; p, q)$ and $\kappa(k, p, q)$ in terms of the regularized coefficients $c(p)$ and $s(p)$.

$$\begin{aligned} J(k; p, q) &= \sqrt{8\omega_{\text{bg}}(p)\omega_{\text{bg}}(q)\omega_{\text{bg}}(k)} j(k; p, q) \\ &= -\hat{v}(k)(c(k) - s(k))(c(p)s(q) + c(q)s(p)) \\ &\quad + \hat{v}(p)(c(p) - s(p))(c(q)c(k) + s(q)s(k)) \\ &\quad + \hat{v}(q)(c(q) - s(q))(c(p)c(k) + s(p)s(k)) \end{aligned} \quad (6.40)$$

and

$$\begin{aligned} K(k, p, q) &= \sqrt{8\omega_{\text{bg}}(p)\omega_{\text{bg}}(q)\omega_{\text{bg}}(k)} \kappa(k, p, q) \\ &= -\hat{v}(k)(c(k) - s(k))(c(p)s(q) + c(q)s(p)) \\ &\quad - \hat{v}(p)(c(p) - s(p))(s(q)c(k) + c(q)s(k)) \\ &\quad - \hat{v}(q)(c(q) - s(q))(s(p)c(k) + c(p)s(k)). \end{aligned} \quad (6.41)$$

From now on we will employ the (t, s) coordinates 3.3.

Expansions for J and K . Since J and K are even functions of s , and since the potential is twice differentiable (3.1c), we obtain

$$J(k; t + sk, t - sk) = J(k; t, t) + k^2 s^2 S_J(k, t, sk), \quad (6.42)$$

$$K(k, t + sk, t - sk) = K(k, t, t) + k^2 s^2 S_K(k, t, sk), \quad (6.43)$$

where the remainders $S_{J/K}$ are defined as

$$S_J(k, t, sk) = \frac{1}{2} \int_{-1}^1 dv (1 - |v|) \left[J(k, t + \cdot, t - \cdot) \right]'' \Big|_{vsk}, \quad (6.44)$$

$$S_K(k, t, sk) = \frac{1}{2} \int_{-1}^1 dv (1 - |v|) \left[K(k, t + \cdot, t - \cdot) \right]'' \Big|_{vsk}. \quad (6.45)$$

If we were to set $k = 0$, we would obtain

$$S_J(0; t, 0) = 8 \left(\frac{\hat{v}'(t) \omega_{\text{bg}}(t)}{t} + \frac{\hat{v}(t) \omega_{\text{bg}}(t)}{t^2} - \frac{\omega'_{\text{bg}}(t) \hat{v}(t)}{t} \right) + 2\hat{v}''(t) \omega_{\text{bg}}(t) + 2\hat{v}(t) \omega''_{\text{bg}}(t) - 4\omega'_{\text{bg}}(t) \hat{v}'(t) \quad (6.46)$$

$$S_K(0, t, 0) = -S_J(0; t, 0). \quad (6.47)$$

We now derive estimates for these remainders

Lemma 6.6. *Suppose that the potential satisfies Assumptions 2.1 and 3.1. Let us introduce the function*

$$\Phi(t) := |\hat{v}(t)| + t|\hat{v}'(t)| + t^2|\hat{v}''(t)| \quad (6.48)$$

Then, we can bound the remainders S_J and S_K as

$$|S_J(k, t, sk)| + |S_K(k, t, sk)| \leq C \int_{-1}^1 dv (1 - |v|) \Phi(t + vsk) \quad (6.49)$$

for k in a sufficiently small neighborhood of 0. In particular,

$$|S_J(k, t, sk)| + |S_K(k, t, sk)| \leq C(1 + t^2) \quad (6.50)$$

for some sufficiently large $C > 0$.

We next examine the diagonal expressions $J(k; t, t)$ and $K(k, t, t)$. The former can be written as

$$J(k; t, t) = 2\hat{v}(t) \left[(2\omega_{\text{bg}}(t) - \omega_{\text{bg}}(k)) \left(1 + \frac{\omega_{\text{bg}}(k)}{4} \frac{t^2}{\omega_{\text{bg}}(t)} \right) + R_J(k, t) \right] \quad (6.51)$$

where

$$R_J(k, t) = \frac{\omega_{\text{bg}}(k)^2}{4} \frac{t^2}{\omega_{\text{bg}}(t)} + 2\omega_{\text{bg}}(t) \left(\frac{\omega_{\text{bg}}(k)^2}{8} + \frac{r(k)}{2} \right) + t(c(t)R_c(k) + s(t)R_s(k)) + (R_s(k) - R_c(k)) - r(k)k. \quad (6.52)$$

Similarly, we have for $K(k, t, t)$

$$K(k, t, t) = -2\hat{v}(t) \left[(2\omega_{\text{bg}}(t) + \omega_{\text{bg}}(k)) \left(1 - \frac{\omega_{\text{bg}}(k)}{4} \frac{t^2}{\omega_{\text{bg}}(t)} \right) + R_K(k, t) \right], \quad (6.53)$$

where

$$R_K(k, t) = \frac{\omega_{\text{bg}}(k)^2}{4} \frac{t^2}{\omega_{\text{bg}}(t)} + 2\omega_{\text{bg}}(t) \left(\frac{\omega_{\text{bg}}(k)^2}{8} + \frac{r(k)}{2} \right) + t(c(t)R_s(k) + s(t)R_c(k)) - (R_s(k) - R_c(k)) + r(k)k. \quad (6.54)$$

The sum and difference of the remainders R_J and R_K are given by

$$R_J + R_K = \frac{\omega_{\text{bg}}(k)^2}{2} \frac{t^2}{\omega_{\text{bg}}(t)} + \omega_{\text{bg}}(t) \left(\frac{\omega_{\text{bg}}(k)^2}{2} + 2r(k) \right) + 2\omega_{\text{bg}}(t)(R_c(k) + R_s(k)), \quad (6.55)$$

$$R_J - R_K = (t^2 - 2)(R_c(k) - R_s(k)) - 2r(k)k. \quad (6.56)$$

Using Equation (6.37) we can also write $J(k; t, t)$ and $K(k, t, t)$ in a more useful form:

$$2\hat{v}(t) \left[(\omega_{\text{bg}}(t + sk) + \omega_{\text{bg}}(t - sk) - \omega_{\text{bg}}(k)) \left(1 + \frac{\omega_{\text{bg}}(k)}{4} \frac{t^2}{\omega_{\text{bg}}(t)} \right) - s^2 k^2 R_\omega(t, sk) \left(1 + \frac{\omega_{\text{bg}}(k)}{4} \frac{t^2}{\omega_{\text{bg}}(t)} \right) + R_J(k, t) \right] = J(k; t, t), \quad (6.57)$$

and

$$-2\hat{v}(t) \left[(\omega_{\text{bg}}(t + sk) + \omega_{\text{bg}}(t - sk) + \omega_{\text{bg}}(k)) \left(1 - \frac{\omega_{\text{bg}}(k)}{4} \frac{t^2}{\omega_{\text{bg}}(t)} \right) - s^2 k^2 R_\omega(t, sk) \left(1 - \frac{\omega_{\text{bg}}(k)}{4} \frac{t^2}{\omega_{\text{bg}}(t)} \right) + R_K(k, t) \right] = K(k, t, t), \quad (6.58)$$

where in both equations we have added and subtracted a term $s^2 k^2 R_\omega(t, sk)$.

R_J and R_K satisfy the following upper bound:

Lemma 6.7. *Suppose that the potential satisfies Assumptions 2.1 and 3.1. Then, we have*

$$\max\{|R_J(k, t)|, |R_K(k, t)|\} \leq Ck^2(1 + t^2), \quad (6.59)$$

for some sufficiently large $C > 0$, as $k \rightarrow 0$.

Density function h . We further define

$$h(t) := \frac{t}{\omega_{\text{bg}}(t)}. \quad (6.60)$$

This function appears naturally as the density factor in the Fermi golden rule integrals after the change of variables in Definition 3.3. We collect its relevant properties below

Lemma 6.8. *Suppose that the potential satisfies Assumptions 2.1 and 3.1. Then $h(t)$, $h'(t)$ and $h''(t)$ are continuous and bounded functions. Moreover, they satisfy*

$$h(t) = O\left(\frac{1}{t}\right), \quad h'(t) = O\left(\frac{1}{t^2}\right), \quad h''(t) = O\left(\frac{1}{t^3}\right) \text{ as } t \rightarrow +\infty. \quad (6.61)$$

It will be useful to have the following expansion at hand

$$h(t + sk)h(t - sk) = h(t)^2 + s^2 k^2 R_h(t, sk), \quad (6.62)$$

$$R_h(t, sk) = \frac{1}{2} \int_{-1}^1 dv (1 - |v|) (h''(t - vsk)h(t + vsk) + h''(t + vsk)h(t - vsk) - 2h'(t - vsk)h'(t + vsk)). \quad (6.63)$$

As a corollary to Lemma 6.8 we get

Corollary 6.9. *Suppose that the potential satisfies Assumptions 2.1 and 3.1. Then*

$$|R_h(t, sk)| \leq C \min\{1, t^{-4}\} \quad (6.64)$$

for some sufficiently large constant $C > 0$, $t \in [\frac{k}{2}, +\infty[$.

Setting $k = 0$ in (6.63) we get

$$R_h(t, 0) = \frac{\omega'_{\text{bg}}(t)^2 t^2 - \omega''_{\text{bg}}(t) \omega_{\text{bg}}(t) t^2 - \omega_{\text{bg}}(t)^2}{\omega_{\text{bg}}(t)^4} \quad (6.65)$$

6.3. Expansions in the energy coordinates. Equations (2.1c), (3.1b) and (3.1a) imply that $p \mapsto \omega_{\text{bg}}(p)$ is invertible in a sufficiently small neighborhood U of the origin. We denote its inverse by $\omega \mapsto p(\omega)$ (or $\omega \mapsto q(\omega)$). As $p, q, k \rightarrow 0$, we can therefore express $j(k; p, q)$ and $\kappa(k, p, q)$ in terms of the corresponding Bogoliubov energies. We introduce the following change of variables

Definition 6.10. $x - y$ coordinates:

$$\begin{aligned} x &:= \omega_{\text{bg}}(p) + \omega_{\text{bg}}(q), & y &= \frac{\omega_{\text{bg}}(p) - \omega_{\text{bg}}(q)}{k}, \\ \omega_{\text{bg}}(p) &= \frac{x + yk}{2}, & \omega_{\text{bg}}(q) &= \frac{x - yk}{2}. \end{aligned} \quad (6.66)$$

With a slight abuse of notation, we use the same symbols J and K for the corresponding functions of the energy variables. Thus, we write

$$J\left(k; p\left(\frac{x + yk}{2}\right), q\left(\frac{x - yk}{2}\right)\right) = J(k; x + yk, x - yk). \quad (6.67)$$

We then obtain the following expansions, whose proof is essentially contained in [23, Sec.3.2].

Lemma 6.11. *Suppose that the potential satisfies Assumptions 2.1 and 3.1. Then, the following expansions hold*

$$\begin{aligned} J(k; x + yk, x - yk) &= 2(x - \omega_{\text{bg}}(k)) + R(k; x + yk, x - yk), \\ &\text{as } x, k \rightarrow 0, \end{aligned} \quad (6.68)$$

$$\begin{aligned} K(k, x + yk, x - yk) &= -2(x + \omega_{\text{bg}}(k)) + S(k, x + yk, x - yk), \\ &\text{as } x, k \rightarrow 0, \end{aligned} \quad (6.69)$$

where R and S are at least C^2 in a neighborhood of $(x, k) = (0, 0)$ and are of order $O\left((x^2 + y^2 k^2)^{\frac{3}{2}}\right)$.

We shall also introduce the following

$$f(\omega) := \frac{d[p(\omega)^2]}{d[\omega^2]} = \left(\frac{\hat{v}(p(\omega))}{\hat{v}(0)} + \frac{1}{2} \left(1 + \frac{1}{\hat{v}(0)p(\omega)} \hat{v}'(p(\omega)) \right) p(\omega)^2 \right)^{-1}. \quad (6.70)$$

As a consequence of Assumptions (2.1c), (3.1a) and (3.1b), f is bounded in a neighborhood of the origin and it satisfies

$$f(\omega) = (1 + O(\omega^2)) \quad (6.71)$$

as $\omega \rightarrow 0$.

6.4. Integration domain. In this subsection, we partition $\mathbb{R}^3$ as the disjoint union of three domains $(I_{\mathbf{k}, i})_{i=1}^3$. These will depend on the momentum vector $\mathbf{k}$. However, for sufficiently small $k = |\mathbf{k}|$, their qualitative properties remain unchanged. We also describe their representations in the energy coordinates (x, y) introduced in (6.66) and in the momentum coordinates (t, s) introduced in (3.35).

We take

$$I_{\mathbf{k},1} := \left\{ \mathbf{p} \in \mathbb{R}^3 \mid 2\omega_{\text{bg}}\left(\frac{k}{2}\right) \leq \omega_{\text{bg}}(p) + \omega_{\text{bg}}(q) < 2\omega_{\text{bg}}(k) - 2\omega_{\text{bg}}\left(\frac{k}{2}\right) \right\}, \quad (6.72)$$

$$I_{\mathbf{k},2} := \left\{ \mathbf{p} \in \mathbb{R}^3 \mid 2\omega_{\text{bg}}(k) - 2\omega_{\text{bg}}\left(\frac{k}{2}\right) \leq \omega_{\text{bg}}(p) + \omega_{\text{bg}}(q), \quad p + q < 4k \right\}, \quad (6.73)$$

$$I_{\mathbf{k},3} := \{ \mathbf{p} \in \mathbb{R}^3 \mid 4k \leq p + q \leq +\infty \}. \quad (6.74)$$

Note that in $I_{\mathbf{k},1}$, the range of $\omega_{\text{bg}}(p) + \omega_{\text{bg}}(q)$ is symmetric with respect to $\omega_{\text{bg}}(k)$. It is clear from their definitions that these sets are pairwise disjoint.

Remark 6.12. As will become clear from the estimate of the Fermi golden rule contribution, the decomposition into the domains $I_{\mathbf{k},1}$, $I_{\mathbf{k},2}$ and $I_{\mathbf{k},3}$ is tailored to the coordinate system best suited to each region:

$I_{\mathbf{k},1}$: The symmetric region around the pole of the Fermi golden rule integral. Energy coordinates are convenient here because they resolve the behavior near the pole.

$I_{\mathbf{k},2}$: The transition region between the near-pole domain, whose boundary is at distance $O(k^3)$ from the pole, and the large-momentum region, which is at distance $O(k)$ from the pole. This region is still best treated in energy coordinates.

$I_{\mathbf{k},3}$: The large-momentum region. In this case the boundary of the integration domain is simpler in momentum variables.

The following lemma describes the minimum of $\omega_{\text{bg}}(p) + \omega_{\text{bg}}(q)$ and its distance from $\omega_{\text{bg}}(k)$.

Lemma 6.13. *Fix $\hat{v}(0) = 1$ and $\nu = 1$. Suppose that the potential satisfies Assumptions 2.1 and 3.1. Then, we have*

$$\min_{\mathbf{p} \in \mathbb{R}^3} (\omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p} - \mathbf{k})) = 2\omega_{\text{bg}}\left(\frac{\mathbf{k}}{2}\right) < \omega_{\text{bg}}(\mathbf{k}). \quad (6.75)$$

Moreover there exist sufficiently large positive constants $D \geq C > 0$, such that for all sufficiently small $k > 0$ one has

$$Ck^3 \leq \omega_{\text{bg}}(k) - 2\omega_{\text{bg}}\left(\frac{k}{2}\right) \leq Dk^3 \quad (6.76)$$

Proof. Equation (6.75) was proved in [21] using the strict convexity of $p \rightarrow \omega_{\text{bg}}(p)$ for sufficiently small p . Under our assumptions, strict convexity follows from Equation (3.1e). Expanding $\omega_{\text{bg}}(k)$ at the origin gives

$$\omega_{\text{bg}}(k) - 2\omega_{\text{bg}}\left(\frac{k}{2}\right) = \frac{3}{16} \left(\hat{v}''(0) + \frac{1}{2} \right) k^3 + O(k^5) \quad (6.77)$$

□

Lemma 6.13 immediately implies that the three domains cover the full integration region

Corollary 6.14. *Suppose that the potential satisfies Assumptions 2.1 and 3.1. Then, for all sufficiently small k ,*

$$I_{\mathbf{k},1} \cup I_{\mathbf{k},2} \cup I_{\mathbf{k},3} = \mathbb{R}^3. \quad (6.78)$$

Proof. $I_{\mathbf{k},1}$. Let us parametrize the domain $I_{\mathbf{k},1}$ in terms of the variables (x, y) in (6.66). The constraint in (6.72) bounds x as

$$0 < 2\omega_{\text{bg}}\left(\frac{k}{2}\right) \leq x < 2\omega_{\text{bg}}(k) - 2\omega_{\text{bg}}\left(\frac{k}{2}\right). \quad (6.79)$$

Using the small-momentum expansion of the dispersion relation, we find

$$ky = \omega_{\text{bg}}(p) - \omega_{\text{bg}}(q) = (p - q) + O(p^3 + q^3), \text{ as } p, q \rightarrow 0, \quad (6.80)$$

and taking into account the symmetry $y \leftrightarrow -y$, we have

$$-\eta_k(x) \leq y \leq \eta_k(x), \quad [-\eta_k(x), \eta_k(x)] \subset [-1 - O(k^2), 1 + O(k^2)] \quad (6.81)$$

for some continuous function η_k , uniformly for $x \in [2\omega_{\text{bg}}(\frac{k}{2}), 2\omega_{\text{bg}}(k) - 2\omega_{\text{bg}}(\frac{k}{2})]$.

Moreover, the endpoints satisfy

$$p\left(\frac{x + k\eta_k(x)}{2}\right) + p\left(\frac{x - k\eta_k(x)}{2}\right) = k, \quad 2\omega_{\text{bg}}\left(\frac{k}{2}\right) \leq x \leq \omega_{\text{bg}}(k), \quad (6.82)$$

$$p\left(\frac{x + k\eta_k(x)}{2}\right) - p\left(\frac{x - k\eta_k(x)}{2}\right) = k, \quad \omega_{\text{bg}}(k) \leq x, \quad (6.83)$$

and

$$\eta_k(\omega_{\text{bg}}(k)) = \frac{\omega_{\text{bg}}(k)}{k}, \quad \eta_k\left(2\omega_{\text{bg}}\left(\frac{k}{2}\right)\right) = 0. \quad (6.84)$$

As $p \rightarrow \omega_{\text{bg}}(p)$ is strictly increasing and convex in a neighborhood of 0, $\omega \rightarrow p(\omega)$ is strictly increasing and concave. Then, we have for $y > 0$ that

$$\partial_y \left(p\left(\frac{x + ky}{2}\right) + p\left(\frac{x - ky}{2}\right) \right) < 0, \quad (6.85)$$

$$\partial_y \left(p\left(\frac{x + ky}{2}\right) - p\left(\frac{x - ky}{2}\right) \right) > 0, \quad (6.86)$$

in $2\omega_{\text{bg}}(\frac{k}{2}) < x < \omega_{\text{bg}}(k)$ and $\omega_{\text{bg}}(k) < x$ respectively. Thus, by the implicit function theorem $x \rightarrow \eta_k(x)$ is C^5 in these intervals. At $x = \omega_{\text{bg}}(k)$, $\eta_k(x)$ is continuous but it has different left and right derivatives, namely

$$\lim_{\epsilon \rightarrow 0} \frac{\eta_k(\omega_{\text{bg}}(k)) - \eta_k(\omega_{\text{bg}}(k) - \epsilon)}{\epsilon} = \frac{p'(\omega_{\text{bg}}(k)) + p'(0)}{k(p'(0) - p'(\omega_{\text{bg}}(k)))} \quad (6.87)$$

and

$$\lim_{\epsilon \rightarrow 0} \frac{\eta_k(\omega_{\text{bg}}(k) + \epsilon) - \eta_k(\omega_{\text{bg}}(k))}{\epsilon} = \frac{p'(0) - p'(\omega_{\text{bg}}(k))}{k(p'(0) + p'(\omega_{\text{bg}}(k)))}. \quad (6.88)$$

$I_{\mathbf{k},2}$. The constraint in $I_{\mathbf{k},2}$ puts an upper bound on x

$$2\omega_{\text{bg}}(k) - 2\omega_{\text{bg}}\left(\frac{k}{2}\right) < x \leq \sup_{p+q < 4k} (\omega_{\text{bg}}(p) + \omega_{\text{bg}}(q)). \quad (6.89)$$

by expanding the dispersion relation as

$$\begin{aligned} \omega_{\text{bg}}(p) + \omega_{\text{bg}}(q) &= (p + q) (1 + O(p^2 + q^2)) \\ &= \omega_{\text{bg}}(p + q) (1 + O(p^2 + q^2)) \text{ as } p, q \rightarrow 0 \end{aligned} \quad (6.90)$$

one gets

$$\sup_{p+q < 4k} (\omega_{\text{bg}}(p) + \omega_{\text{bg}}(q)) = \omega_{\text{bg}}(4k) (1 + O(k^2)). \quad (6.91)$$

Therefore, the y integration region is of the form

$$-1 - O(k^2) \leq y \leq 1 + O(k^2) \quad (6.92)$$

for all

$$x \in \left[2\omega_{\text{bg}}(k) - 2\omega_{\text{bg}}\left(\frac{k}{2}\right), \sup_{p+q < 4k} (\omega_{\text{bg}}(p) + \omega_{\text{bg}}(q)) \right],$$

where the remainder term $O(k^2)$ in the integration region possibly depends on x .

$I_{\mathbf{k},3}$. Finally, we observe that in terms of the variables (t, s, φ) , the domain $I_{\mathbf{k},3}$ simply becomes

$$I_{\mathbf{k},3} = [2k, +\infty[\times \left[-\frac{1}{2}, \frac{1}{2} \right] \times [0, 2\pi[. \quad (6.93)$$

□

We conclude this subsection by locating the pole of the first Fermi golden rule integral (3.29) and (3.30)

Proposition 6.15. *Suppose that the potential satisfies Assumptions 2.1 and 3.1. Then, for all sufficiently small k , the equation*

$$x - \omega_{\text{bg}}(k) = \omega_{\text{bg}}(p) + \omega_{\text{bg}}(q) - \omega_{\text{bg}}(k) = 0, \quad (6.94)$$

does not have any solutions if $p > k$.

Proof. By our Assumptions on the potential (2.1c) and (3.1b), we know that there exists a neighborhood of $p = 0$, say $U = [0, c[$ for some $c > 0$, such that $p \rightarrow \omega_{\text{bg}}(p)$ is strictly increasing. Moreover, by (2.1c) we also know that

$$\inf_{p > c} \omega_{\text{bg}}(p) > 0. \quad (6.95)$$

We can take $k < c$ sufficiently small so that

$$\omega_{\text{bg}}(k) < \inf_{p > c} \omega_{\text{bg}}(p). \quad (6.96)$$

Therefore, if $p > k$, we have

$$\omega_{\text{bg}}(p) + \omega_{\text{bg}}(q) > \omega_{\text{bg}}(p) > \min\left\{\inf_{k < p < c} \omega_{\text{bg}}(p), \inf_{p > c} \omega_{\text{bg}}(p)\right\} = \omega_{\text{bg}}(k). \quad (6.97)$$

□

6.5. Estimates for the energy integrals. We take k sufficiently small that Corollary 6.14 applies. We write

$$\begin{aligned} \delta(I_{\mathbf{k},i}) = & -\frac{1}{2}\text{Re} \int_{I_{\mathbf{k},i}} \frac{d\mathbf{p}}{(2\pi)^3} \left[\frac{j(\mathbf{k}; \mathbf{p}, \mathbf{p} - \mathbf{k})^2}{\omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p} - \mathbf{k}) - \omega_{\text{bg}}(\mathbf{k}) - i0} \right. \\ & \left. + \frac{\kappa(\mathbf{k}, \mathbf{p}, \mathbf{p} - \mathbf{k})^2}{\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p} - \mathbf{k})} \right]. \end{aligned} \quad (6.98)$$

In the energy coordinates of Definition 6.10,

$$d\mathbf{p} = \frac{x^2 - k^2 y^2}{8} f\left(\frac{x + ky}{2}\right) f\left(\frac{x - ky}{2}\right) dx dy d\varphi. \quad (6.99)$$

Estimate of $\delta(I_{\mathbf{k},1})$.

Proposition 6.16. *Under Assumptions 2.1 and 3.1,*

$$\delta(I_{\mathbf{k},1}) = \hat{v}(0)\nu^{3/2}O\left(\frac{k^3}{\nu^{3/2}}\right), \quad \frac{k}{\sqrt{\nu}} \rightarrow 0.$$

Proof. We first set $\nu = \hat{v}(0) = 1$, as in Remark 6.1. After the angular integration, the contribution containing J is

$$-\frac{1}{128\pi^2\omega_{\text{bg}}(k)}\text{Re} \int_{2\omega_{\text{bg}}(k/2)}^{2\omega_{\text{bg}}(k)-2\omega_{\text{bg}}(k/2)} \frac{F_k(x)}{x - \omega_{\text{bg}}(k) - i0} dx, \quad (6.100)$$

where, using the same endpoint function η_k ,

$$\begin{aligned} G(k, x, y) &:= f\left(\frac{x + ky}{2}\right) f\left(\frac{x - ky}{2}\right) J(k; x + ky, x - ky)^2, \\ F_k(x) &:= \int_{-\eta_k(x)}^{\eta_k(x)} G(k, x, y) dy. \end{aligned} \quad (6.101)$$

On $I_{\mathbf{k},1}$, Lemmas 6.13 and 6.11, and local regularity, give

$$\begin{aligned} ck^3 &\leq \omega_{\text{bg}}(k) - 2\omega_{\text{bg}}(k/2) \leq Ck^3, & 0 &\leq \eta_k(x) \leq C, \\ |J(k; x + ky, x - ky)| &\leq Ck^3, & |\partial_x J(k; x + ky, x - ky)| &\leq C, \\ |G(k, x, y)| &\leq Ck^6, & |\partial_x G(k, x, y)| &\leq Ck^3, & |F_k(x)| &\leq Ck^6. \end{aligned}$$

Here the derivative bound follows from the smooth expression for J ; also f and f' are bounded near zero.

We only need to differentiate F_k near the pole. For

$$\frac{\omega_{\text{bg}}(k) + 2\omega_{\text{bg}}(k/2)}{2} \leq x < \omega_{\text{bg}}(k),$$

the endpoint relation $p + q = k$, Taylor's formula and $\omega_{\text{bg}}''(p) = O(p)$ imply

$$ck^3 \leq x - 2\omega_{\text{bg}}(k/2) \leq Ck(p - q)^2 \leq Ck^3\eta_k(x)^2, \quad \eta_k(x) \geq c.$$

The inverse dispersion satisfies

$$\begin{aligned} p'(\omega) &= 1 + O(\omega^2), & p''(\omega) &= -\frac{3}{4}(1 + 2\hat{v}''(0))\omega + O(\omega^3), \\ p'\left(\frac{x - k\eta_k(x)}{2}\right) &- p'\left(\frac{x + k\eta_k(x)}{2}\right) &\geq c x k \eta_k(x) &\geq ck^2. \end{aligned}$$

Implicit differentiation of the endpoint relation therefore gives

$$|\eta'_k(x)| = \frac{p'\left(\frac{x + k\eta_k(x)}{2}\right) + p'\left(\frac{x - k\eta_k(x)}{2}\right)}{k \left| p'\left(\frac{x + k\eta_k(x)}{2}\right) - p'\left(\frac{x - k\eta_k(x)}{2}\right) \right|} \leq Ck^{-3}.$$

For $x > \omega_{\text{bg}}(k)$ in $I_{\mathbf{k},1}$ the endpoint relation is $p - q = k$, and instead

$$|\eta'_k(x)| = \frac{p'\left(\frac{x - k\eta_k(x)}{2}\right) - p'\left(\frac{x + k\eta_k(x)}{2}\right)}{k \left[p'\left(\frac{x - k\eta_k(x)}{2}\right) + p'\left(\frac{x + k\eta_k(x)}{2}\right) \right]} \leq Ck.$$

Consequently,

$$\begin{aligned} |F'_k(x)| &\leq \int_{-\eta_k(x)}^{\eta_k(x)} |\partial_x G(k, x, y)| dy + 2|\eta'_k(x)| \sup_{|y| \leq \eta_k(x)} |G(k, x, y)| \leq Ck^3, \\ 0 < |x - \omega_{\text{bg}}(k)| &\leq \frac{\omega_{\text{bg}}(k) - 2\omega_{\text{bg}}(k/2)}{2}. \end{aligned}$$

Since F_k is continuous at the pole, symmetrization gives

$$\begin{aligned} &\text{Re} \int_{2\omega_{\text{bg}}(k/2)}^{2\omega_{\text{bg}}(k) - 2\omega_{\text{bg}}(k/2)} \frac{F_k(x)}{x - \omega_{\text{bg}}(k) - i\epsilon} dx \\ &= \int_0^{\omega_{\text{bg}}(k) - 2\omega_{\text{bg}}(k/2)} \frac{r [F_k(\omega_{\text{bg}}(k) + r) - F_k(\omega_{\text{bg}}(k) - r)]}{r^2 + \epsilon^2} dr. \end{aligned} \quad (6.102)$$

The right-hand side is bounded, uniformly for $\epsilon > 0$, by

$$Ck^3 \int_0^{(\omega_{\text{bg}}(k) - 2\omega_{\text{bg}}(k/2))/2} dr + Ck^6 \int_{(\omega_{\text{bg}}(k) - 2\omega_{\text{bg}}(k/2))/2}^{\omega_{\text{bg}}(k) - 2\omega_{\text{bg}}(k/2)} \frac{dr}{r} \leq Ck^6.$$

This also supplies an integrable majorant for the limit $\epsilon \downarrow 0$. Thus (6.100) is $O(k^5)$.

For the second channel, $K(k; x + ky, x - ky) = O(k)$ and $x + \omega_{\text{bg}}(k) \geq ck$ on $I_{\mathbf{k},1}$. Hence

$$\begin{aligned} & \frac{1}{128\pi^2\omega_{\text{bg}}(k)} \int_{2\omega_{\text{bg}}(k/2)}^{2\omega_{\text{bg}}(k)-2\omega_{\text{bg}}(k/2)} dx \int_{-\eta_k(x)}^{\eta_k(x)} dy \\ & \quad \times f\left(\frac{x+ky}{2}\right) f\left(\frac{x-ky}{2}\right) \frac{K(k; x+ky, x-ky)^2}{x+\omega_{\text{bg}}(k)} \\ & \leq \frac{C}{k} k^3 \frac{k^2}{k} = Ck^3. \end{aligned} \quad (6.103)$$

Restoring the factor $\hat{v}(0)\nu^{3/2}$ and replacing k by $\frac{k}{\sqrt{\nu}}$ proves the claim. $\square$

Estimate of $\delta(I_{\mathbf{k},2})$.

Proposition 6.17. *Under Assumption 2.1 and 3.1,*

$$\delta(I_{\mathbf{k},2}) = -\frac{15\hat{v}(0)\nu^{3/2}}{16\pi^2} \frac{k}{\sqrt{\nu}} \left(1 + O\left(\frac{k^2}{\nu}\right)\right), \quad \frac{k}{\sqrt{\nu}} \rightarrow 0.$$

Proof. Again we take $\nu = \hat{v}(0) = 1$. On $I_{\mathbf{k},2}$,

$$x - \omega_{\text{bg}}(k) \geq \omega_{\text{bg}}(k) - 2\omega_{\text{bg}}(k/2) \geq ck^3, \quad (6.104)$$

$$x + \omega_{\text{bg}}(k) \geq ck. \quad (6.105)$$

In particular, there is no pole. Lemma 6.11 gives $R, S = O(k^3)$ and the exact square expansion

$$\begin{aligned} & \frac{J(k; x+ky, x-ky)^2}{x-\omega_{\text{bg}}(k)} + \frac{K(k; x+ky, x-ky)^2}{x+\omega_{\text{bg}}(k)} \\ & = 8x + 4(R(k; x+ky, x-ky) - S(k; x+ky, x-ky)) \\ & \quad + \frac{R(k; x+ky, x-ky)^2}{x-\omega_{\text{bg}}(k)} + \frac{S(k; x+ky, x-ky)^2}{x+\omega_{\text{bg}}(k)} = 8x + O(k^3). \end{aligned} \quad (6.106)$$

We integrate this expression in the momentum coordinates (t, s) . Uniformly for $k/2 \leq t \leq 2k$ and $|s| \leq \frac{1}{2}$,

$$\begin{aligned} x &= \omega_{\text{bg}}(t+ks) + \omega_{\text{bg}}(t-ks) = 2t + O(k^3), \\ h(t+ks)h(t-ks) &= 1 + O(k^2), \quad d\mathbf{p} = 2(t^2 - k^2s^2) d\varphi dt ds. \end{aligned}$$

Thus the momentum integrand on $I_{\mathbf{k},2}$ satisfies

$$h(t+ks)h(t-ks)[8x + O(k^3)] = 16t + O(k^3).$$

Since the lower energy cutoff is $k + O(k^3)$, the actual domain satisfies, after the angular integration,

$$[k/2 + Ck^3, 2k] \times [-\frac{1}{2}, \frac{1}{2}] \subset I_{\mathbf{k},2} \subset [k/2, 2k] \times [-\frac{1}{2}, \frac{1}{2}].$$

The omitted strip has area $O(k^3)$. Its leading integrand is $O(k)$, while the integrated remainder is $O(k^3)O(k) = O(k^4)$. Therefore

$$\begin{aligned} \delta(I_{\mathbf{k},2}) &= -\frac{1}{32\pi^2\omega_{\text{bg}}(k)} \left[\int_{k/2}^{2k} dt \int_{-\frac{1}{2}}^{\frac{1}{2}} ds 16t + O(k^4) \right] \\ &= -\frac{30k^2 + O(k^4)}{32\pi^2\omega_{\text{bg}}(k)} = -\frac{15}{16\pi^2} k + O(k^3). \end{aligned}$$

The rescaling of Remark 6.1 gives the stated result. $\square$

It remains to estimate $\delta(I_{\mathbf{k},3})$. For this purpose it is convenient to use the momentum coordinates introduced in Definition (3.35). With this coordinates, the relevant integral ($\nu = \hat{v}(0) = 1$) becomes

$$\begin{aligned} \delta(I_{\mathbf{k},3}) = & -\frac{1}{\omega_{\text{bg}}(k)} \frac{1}{32\pi^2} \int_{2k}^{+\infty} dt \int_{-\frac{1}{2}}^{\frac{1}{2}} ds \frac{t^2 - k^2 s^2}{\omega_{\text{bg}}(t+sk) \omega_{\text{bg}}(t-sk)} \\ & \times \left(\frac{J(k; t+sk, t-sk)^2}{\omega_{\text{bg}}(t+sk) + \omega_{\text{bg}}(t-sk) - \omega_{\text{bg}}(k)} + \frac{K(k, t+sk, t-sk)^2}{\omega_{\text{bg}}(t+sk) + \omega_{\text{bg}}(t-sk) + \omega_{\text{bg}}(k)} \right) \end{aligned} \quad (6.107)$$

To determine the small $\mathbf{k}$ -behavior of this integral, we expand the last line of Equation (6.107) using (6.57) and (6.58). As a preliminary step, we write it as

$$\frac{J(k; t, t)^2}{\omega_{\text{bg}}(t+sk) + \omega_{\text{bg}}(t-sk) - \omega_{\text{bg}}(k)} + \frac{K(k, t, t)^2}{\omega_{\text{bg}}(t+sk) + \omega_{\text{bg}}(t-sk) + \omega_{\text{bg}}(k)} \quad (6.108)$$

$$+ S_{\text{shift}}(k, t, sk) + R_3(k, t, sk), \quad (6.109)$$

where we have defined the *shift* term

$$S_{\text{shift}}(k, t, sk) := 4k^2 s^2 \hat{v}(t) (S_J(k, t, sk) - S_K(k, t, sk)). \quad (6.110)$$

Expanding the squares in (6.42) and (6.43), and then using (6.57) and (6.58), gives the exact remainder

$$\begin{aligned} R_3(k, t, sk) = & \frac{\hat{v}(t) \omega_{\text{bg}}(k) t^2}{\omega_{\text{bg}}(t)} k^2 s^2 (S_J(k, t, sk) + S_K(k, t, sk)) \\ & + \frac{4\hat{v}(t) k^2 s^2 S_J(k, t, sk)}{\omega_{\text{bg}}(t+sk) + \omega_{\text{bg}}(t-sk) - \omega_{\text{bg}}(k)} \\ & \times \left[R_J(k, t) - s^2 k^2 R_\omega(t, sk) \left(1 + \frac{\omega_{\text{bg}}(k) t^2}{4\omega_{\text{bg}}(t)} \right) \right] \\ & + \frac{k^4 s^4 S_J(k, t, sk)^2}{\omega_{\text{bg}}(t+sk) + \omega_{\text{bg}}(t-sk) - \omega_{\text{bg}}(k)} \end{aligned} \quad (6.111)$$

$$\begin{aligned} & + \frac{4\hat{v}(t) k^2 s^2 S_K(k, t, sk)}{\omega_{\text{bg}}(t+sk) + \omega_{\text{bg}}(t-sk) + \omega_{\text{bg}}(k)} \\ & \times \left[s^2 k^2 R_\omega(t, sk) \left(1 - \frac{\omega_{\text{bg}}(k) t^2}{4\omega_{\text{bg}}(t)} \right) - R_K(k, t) \right] \\ & + \frac{k^4 s^4 S_K(k, t, sk)^2}{\omega_{\text{bg}}(t+sk) + \omega_{\text{bg}}(t-sk) + \omega_{\text{bg}}(k)}. \end{aligned} \quad (6.112)$$

To extract the same infrared term as in the Hellmann–Feynman contribution, we first use $c(k) + s(k) = 2\frac{\omega_{\text{bg}}(k)}{k}$ in (6.55). This gives the exact identity

$$R_J(k, t) + R_K(k, t) = \frac{\omega_{\text{bg}}(k)^2 t^2}{2\omega_{\text{bg}}(t)} + 4\omega_{\text{bg}}(t) \left(\frac{\omega_{\text{bg}}(k)}{k} - 1 \right).$$

Together with

$$2\frac{\omega_{\text{bg}}(k)}{k} - 1 = \frac{\omega_{\text{bg}}(k)^2}{k^2} - \left(\frac{\omega_{\text{bg}}(k)}{k} - 1 \right)^2,$$

this allows us to absorb the entire sum $R_J + R_K$ into the main term and a remainder containing $\left(\frac{\omega_{\text{bg}}(k)}{k} - 1 \right)^2$. The diagonal contribution (6.108) is then decomposed exactly as

$$(6.108) = S_{\text{main}}(k, t) \quad (6.113)$$

$$- 8\hat{v}(t)^2 s^2 k^2 R_\omega(t, sk) + S_3(k, t, sk), \quad (6.114)$$

where

$$S_{\text{main}}(k, t) := 16\hat{v}(t)^2\omega_{\text{bg}}(t)\frac{\omega_{\text{bg}}(k)^2}{k^2} + \frac{\hat{v}(t)^2\omega_{\text{bg}}(k)^2t^4}{\omega_{\text{bg}}(t)} \quad (6.115)$$

and

$$\begin{aligned} S_3(k, t, sk) = & \frac{2\hat{v}(t)^2t^2}{\omega_{\text{bg}}(t)}\omega_{\text{bg}}(k)(R_J(k, t) - R_K(k, t)) \\ & - \frac{\hat{v}(t)^2t^4R_\omega(t, sk)}{2\omega_{\text{bg}}(t)^2}s^2\omega_{\text{bg}}(k)^2k^2 \\ & - 16\hat{v}(t)^2\omega_{\text{bg}}(t)\left(\frac{\omega_{\text{bg}}(k)}{k} - 1\right)^2 \end{aligned} \quad (6.116)$$

$$+ 4\hat{v}(t)^2\frac{\left[R_J(k, t) - s^2k^2R_\omega(t, sk)\left(1 + \frac{\omega_{\text{bg}}(k)t^2}{4\omega_{\text{bg}}(t)}\right)\right]^2}{\omega_{\text{bg}}(t+sk) + \omega_{\text{bg}}(t-sk) - \omega_{\text{bg}}(k)} \quad (6.117)$$

$$+ 4\hat{v}(t)^2\frac{\left[R_K(k, t) - s^2k^2R_\omega(t, sk)\left(1 - \frac{\omega_{\text{bg}}(k)t^2}{4\omega_{\text{bg}}(t)}\right)\right]^2}{\omega_{\text{bg}}(t+sk) + \omega_{\text{bg}}(t-sk) + \omega_{\text{bg}}(k)}. \quad (6.118)$$

Estimate of $\delta(I_{\mathbf{k},3})$. We work in the normalization $\nu = \hat{v}(0) = 1$ and retain Assumptions 2.1 and 3.1.

The boundedness and continuity of $\hat{v}$, its behavior at the origin, and stability imply

$$C^{-1}t(1+t) \leq \omega_{\text{bg}}(t) \leq Ct(1+t), \quad t > 0.$$

Throughout the estimates, $0 < k \leq k_0 < \infty$ is sufficiently small, $t \geq 2k$, and $|s| \leq \frac{1}{2}$. Constants may depend on the fixed potential but are independent of k, t, s .

We shall use

$$\omega_{\text{bg}}(t+sk) + \omega_{\text{bg}}(t-sk) \pm \omega_{\text{bg}}(k) \geq C^{-1}t(1+t), \quad (6.119)$$

$$h(t+sk)h(t-sk) \leq \frac{C}{1+t^2},$$

$$\frac{t^2}{\omega_{\text{bg}}(t)} \leq C \min\{t, 1\}.$$

For small t , the first inequality follows from $\omega_{\text{bg}}(t) = t + O(t^3)$ and $2k \leq t$. Away from the origin, the explicit expression of ω_{bg} give the same bound.

We also recall the exact Taylor formula for the density factor:

$$h(t+sk)h(t-sk) = h(t)^2 + s^2k^2R_h(t, sk), \quad (6.120)$$

where

$$\begin{aligned} R_h(t, sk) = & \frac{1}{2} \int_{-1}^1 dv (1 - |v|) \left[h''(t + vsk)h(t - vsk) \right. \\ & \left. + h(t + vsk)h''(t - vsk) - 2h'(t + vsk)h'(t - vsk) \right]. \end{aligned} \quad (6.121)$$

By Lemma 6.8 and Corollaries 6.5 and 6.9

$$|R_h(t, sk)| \leq C \min\{1, t^{-4}\}, \quad |R_\omega(t, sk)| \leq C. \quad (6.122)$$

The following four lemmas estimate the main term, S_3 , the shift term S_{shift} , and R_3 , respectively.

Lemma 6.18. *Under Assumptions 2.1 and 3.1 with $\nu = \hat{v}(0) = 1$,*

$$-\frac{1}{32\pi^2\omega_{\text{bg}}(k)} \int_{2k}^{\infty} dt \int_{-\frac{1}{2}}^{\frac{1}{2}} ds h(t+sk)h(t-sk) \left[S_{\text{main}}(k, t) - 8\hat{v}(t)^2 s^2 k^2 R_{\omega}(t, sk) \right] \quad (6.123)$$

$$= -c_{\text{ir}}(k, 1, \hat{v}) + k c_{\text{main}}(\hat{v}) + o(k), \quad k \rightarrow 0. \quad (6.124)$$

Here

$$c_{\text{ir}}(k, 1, \hat{v}) = \frac{\omega_{\text{bg}}(k)}{2\pi^2 k^2} \int_0^{\infty} dt \frac{\hat{v}(t)^2 t^2}{\omega_{\text{bg}}(t)}, \quad (6.125)$$

as in (3.24), and

$$c_{\text{main}}(\hat{v}) = \frac{1}{\pi^2} - \int_0^{\infty} \frac{dt}{24\pi^2} \hat{v}(t)^2 \omega_{\text{bg}}(t) R_h(t, 0) - \int_0^{\infty} \frac{dt}{32\pi^2} \frac{\hat{v}(t)^2 t^6}{\omega_{\text{bg}}(t)^3} + \int_0^{\infty} \frac{dt}{48\pi^2} \frac{\hat{v}(t)^2 t^2}{\omega_{\text{bg}}(t)^2} R_{\omega}(t, 0).$$

Proof. The first term in S_{main} gives the exact decomposition

$$-\frac{\omega_{\text{bg}}(k)}{2\pi^2 k^2} \int_{2k}^{\infty} dt \int_{-\frac{1}{2}}^{\frac{1}{2}} ds h(t+sk)h(t-sk) \hat{v}(t)^2 \omega_{\text{bg}}(t) = \frac{\omega_{\text{bg}}(k)}{2\pi^2 k^2} \int_0^{2k} dt \frac{t^2 \hat{v}(t)^2}{\omega_{\text{bg}}(t)} \quad (6.126)$$

$$-\frac{\omega_{\text{bg}}(k)}{2\pi^2} \int_{2k}^{\infty} dt \int_{-\frac{1}{2}}^{\frac{1}{2}} ds s^2 R_h(t, sk) \hat{v}(t)^2 \omega_{\text{bg}}(t) \quad (6.127)$$

$$-c_{\text{ir}}(k, 1, \hat{v}). \quad (6.128)$$

Since $\frac{t^2 \hat{v}(t)^2}{\omega_{\text{bg}}(t)} = t + O(t^3)$ at the origin,

$$(6.126) = \frac{k}{\pi^2} + O(k^3).$$

Moreover,

$$|R_h(t, sk) \hat{v}(t)^2 \omega_{\text{bg}}(t)| \leq C \hat{v}(t)^2 \min\{1, t^{-2}\}.$$

This is an integrable upper bound independent of k, s . Thus dominated convergence, together with $\omega_{\text{bg}}(k) = k + O(k^3)$, yields

$$(6.127) = -\frac{k}{24\pi^2} \int_0^{\infty} dt \hat{v}(t)^2 \omega_{\text{bg}}(t) R_h(t, 0) + o(k).$$

For the second term in S_{main} , use the exact identity

$$\omega_{\text{bg}}(k)^2 \hat{v}(t)^2 h(t+sk)h(t-sk) \frac{t^4}{\omega_{\text{bg}}(t)} = \omega_{\text{bg}}(k)^2 \hat{v}(t)^2 h(t)^2 \frac{t^4}{\omega_{\text{bg}}(t)} \quad (6.129)$$

$$+ \omega_{\text{bg}}(k)^2 s^2 k^2 \hat{v}(t)^2 R_h(t, sk) \frac{t^4}{\omega_{\text{bg}}(t)}. \quad (6.130)$$

The absolute value of (6.130) is bounded by $C k^4 \hat{v}(t)^2 \min\{1, t^{-2}\}$. Its contribution after integration and multiplication by $\omega_{\text{bg}}(k)^{-1}$ is therefore $O(k^3)$. The integrand in (6.129) is $O(t^3)$ at the origin and $O(\hat{v}(t)^2)$ at infinity. Extending its integral from

$[2k, \infty[$ to $[0, \infty[$ therefore changes the energy contribution by $O(k^5)$. Replacing the remaining factor $\omega_{\text{bg}}(k)$ by k introduces an $O(k^3)$ error. Consequently,

$$\begin{aligned} & -\frac{\omega_{\text{bg}}(k)}{32\pi^2} \int_{2k}^{\infty} dt \int_{-\frac{1}{2}}^{\frac{1}{2}} ds \hat{v}(t)^2 h(t+sk)h(t-sk) \frac{t^4}{\omega_{\text{bg}}(t)} \\ & = -\frac{k}{32\pi^2} \int_0^{\infty} dt \frac{\hat{v}(t)^2 t^6}{\omega_{\text{bg}}(t)^3} + O(k^3). \end{aligned}$$

The explicitly retained R_{ω} term is handled by dominated convergence, using $h(t+sk)h(t-sk)|R_{\omega}(t, sk)| \leq C(1+t^2)^{-1}$:

$$\begin{aligned} & \frac{k^2}{4\pi^2\omega_{\text{bg}}(k)} \int_{2k}^{\infty} dt \int_{-\frac{1}{2}}^{\frac{1}{2}} ds s^2 \hat{v}(t)^2 h(t+sk)h(t-sk) R_{\omega}(t, sk) \\ & = \frac{k}{48\pi^2} \int_0^{\infty} dt \frac{\hat{v}(t)^2 t^2}{\omega_{\text{bg}}(t)^2} R_{\omega}(t, 0) + o(k). \end{aligned}$$

Combining these three contributions proves the stated formula for c_{main} . $\square$

Lemma 6.19. *Under Assumptions 2.1 and 3.1 with $\nu = \hat{v}(0) = 1$,*

$$\frac{1}{\omega_{\text{bg}}(k)} \int_{2k}^{\infty} dt \int_{-\frac{1}{2}}^{\frac{1}{2}} ds h(t+sk)h(t-sk) |S_3(k, t, sk)| \leq Ck^3(1+|\log k|) = o(k).$$

Proof. We first estimate the three terms in Equation (6.116). By (6.56), $|R_J(k, t) - R_K(k, t)| \leq Ck^3(1+t^2)$. By corollary 6.5 we have

$$\left| \frac{\hat{v}(t)^2 t^4 R_{\omega}(t, sk)}{2\omega_{\text{bg}}(t)^2} s^2 \omega_{\text{bg}}(k)^2 k^2 \right| \leq Ck^4 \hat{v}(t)^2.$$

Moreover, $\frac{\omega_{\text{bg}}(k)}{k} - 1 = O(k^2)$, so that

$$16\hat{v}(t)^2 \omega_{\text{bg}}(t) \left(\frac{\omega_{\text{bg}}(k)}{k} - 1 \right)^2 \leq Ck^4 \hat{v}(t)^2 (1+t^2).$$

Thus the absolute value of the three terms in (6.116) is at most $Ck^4 \hat{v}(t)^2 (1+t^2)$. Multiplying by the $h(t+sk)h(t-sk)$ and integrating gives an $O(k^3)$ contribution, because $\hat{v} \in L^2(0, \infty)$.

For the two square terms (6.117), (6.118), Lemma 6.7 and the bounds above give

$$\begin{aligned} & \left| R_J(k, t) - s^2 k^2 R_{\omega}(t, sk) \left(1 + \frac{\omega_{\text{bg}}(k)t^2}{4\omega_{\text{bg}}(t)} \right) \right|^2 \leq Ck^4 (1+t^2)^2, \\ & \left| R_K(k, t) - s^2 k^2 R_{\omega}(t, sk) \left(1 - \frac{\omega_{\text{bg}}(k)t^2}{4\omega_{\text{bg}}(t)} \right) \right|^2 \leq Ck^4 (1+t^2)^2. \end{aligned}$$

It follows from the lower bound in (6.119) that

$$\begin{aligned} & \frac{1}{\omega_{\text{bg}}(k)} \int_{2k}^{\infty} dt \int_{-\frac{1}{2}}^{\frac{1}{2}} ds h(t+sk)h(t-sk) |S_3(k, t, sk)| \\ & \leq Ck^3 \int_0^{\infty} \hat{v}(t)^2 dt + Ck^3 \int_{2k}^{\infty} dt \hat{v}(t)^2 \frac{1+t^2}{t(1+t)} \\ & \leq Ck^3 \left(1 + |\log k| + \int_1^{\infty} \hat{v}(t)^2 dt \right). \end{aligned}$$

This proves the assertion. $\square$

Lemma 6.20. *Under Assumptions 2.1 and 3.1 with $\nu = \hat{v}(0) = 1$,*

$$-\frac{1}{32\pi^2\omega_{\text{bg}}(k)} \int_{2k}^{\infty} dt \int_{-\frac{1}{2}}^{\frac{1}{2}} ds h(t+sk)h(t-sk)S_{\text{shift}}(k, t, sk) = kc_{\text{shift}}(\hat{v}) + o(k),$$

where

$$\begin{aligned} c_{\text{shift}}(\hat{v}) &= -\frac{1}{48\pi^2} \int_0^{\infty} dt \frac{\hat{v}(t)t^2}{\omega_{\text{bg}}(t)^2} S_J(0; t, 0) \\ &= -\frac{1}{48\pi^2} \int_0^{\infty} dt \left[\frac{8\hat{v}(t)t^2}{\omega_{\text{bg}}(t)^2} \left(\frac{\hat{v}'(t)\omega_{\text{bg}}(t)}{t} + \frac{\hat{v}(t)\omega_{\text{bg}}(t)}{t^2} - \frac{\omega'_{\text{bg}}(t)\hat{v}(t)}{t} \right) \right. \\ &\quad \left. + \frac{2\hat{v}(t)t^2}{\omega_{\text{bg}}(t)^2} \left(\hat{v}''(t)\omega_{\text{bg}}(t) + \hat{v}(t)\omega''_{\text{bg}}(t) - 2\omega'_{\text{bg}}(t)\hat{v}'(t) \right) \right]. \end{aligned}$$

Proof. We use the function $\Phi(t) = |\hat{v}(t)| + t|\hat{v}'(t)| + t^2|\hat{v}''(t)|$. By Lemma 6.6,

$$\begin{aligned} &h(t+sk)h(t-sk)s^2|\hat{v}(t)S_{J/K}(k, t, sk)| \\ &\leq C|\hat{v}(t)| \min\{1, t^{-2}\} \int_{-1}^1 dv (1-|v|)\Phi(t+vs k). \end{aligned} \quad (6.131)$$

On any interval $[x, \infty[$, $x > 1$, Cauchy–Schwarz bounds, bound the integral of the right-hand side by

$$\begin{aligned} &C \left(\int_x^{\infty} \frac{\hat{v}(t)^2}{t^2} dt \right)^{\frac{1}{2}} \left(\int_x^{\infty} \frac{dt}{t^2} \left[\int_{-1}^1 dv (1-|v|)\Phi(t+vs k) \right]^2 \right)^{\frac{1}{2}} \\ &= C \left(\int_x^{\infty} \frac{\hat{v}(t)^2}{t^2} dt \right)^{\frac{1}{2}} \left(\int_{1-svk}^{\infty} \frac{dt}{(t-svk)^2} \int_{-1}^1 dv (1-|v|)\Phi(t+vs k)^2 \right)^{\frac{1}{2}} \\ &\leq D \left(\int_x^{\infty} \frac{\hat{v}(t)^2}{t^2} dt \right)^{\frac{1}{2}} \left(\int_{\frac{1}{2}}^{\infty} \frac{\Phi(t)^2}{t^2} dt \right)^{\frac{1}{2}} < \infty. \end{aligned} \quad (6.132)$$

where we first used Jensen inequality and then we changed variable $t+vs k \rightarrow t$ in the third one. The first factor in (6.132) goes to zero as $x \rightarrow +\infty$, and both factors are uniformly bounded in s, k .

For each fixed $t > 0$, continuity of $S_{J/K}$ gives

$$S_J(k, t, sk) \rightarrow S_J(0; t, 0), \quad S_K(k, t, sk) \rightarrow S_K(0; t, 0).$$

On every bounded interval, (6.131) is uniformly bounded, including near the origin. Dominated convergence there, together with uniform smallness of the tails in Equation (6.132), proves

$$\begin{aligned} &\lim_{k \rightarrow 0} \int_{2k}^{\infty} dt \int_{-\frac{1}{2}}^{\frac{1}{2}} ds h(t+sk)h(t-sk)s^2\hat{v}(t)S_{J/K}(k, t, sk) \\ &= \frac{1}{12} \int_0^{\infty} dt h(t)^2\hat{v}(t)S_{J/K}(0, t, 0). \end{aligned}$$

Using $S_K(0, t, 0) = -S_J(0; t, 0)$, we consequently obtain

$$\begin{aligned} &\int_{2k}^{\infty} dt \int_{-\frac{1}{2}}^{\frac{1}{2}} ds h(t+sk)h(t-sk)S_{\text{shift}}(k, t, sk) \\ &= \frac{2k^2}{3} \int_0^{\infty} dt h(t)^2\hat{v}(t)S_J(0; t, 0) + o(k^2). \end{aligned}$$

Multiplication by $-\frac{1}{32\pi^2\omega_{\text{bg}}(k)}$ proves the assertion. The explicit formula for c_{shift} follows from (6.46). $\square$

Lemma 6.21. *Under Assumptions 2.1 and 3.1 with $\nu = \hat{v}(0) = 1$,*

$$\frac{1}{\omega_{\text{bg}}(k)} \int_{2k}^{\infty} dt \int_{-\frac{1}{2}}^{\frac{1}{2}} ds h(t+sk)h(t-sk)|R_3(k,t,sk)| \leq Ck^2.$$

Proof. The exact formula (6.111)–(6.112), the estimates on R_J, R_K and R_ω , and (6.119) imply

$$\begin{aligned} |R_3(k,t,sk)| &\leq Ck^3|\hat{v}(t)|\frac{t^2}{\omega_{\text{bg}}(t)}(|S_J(k,t,sk)| + |S_K(k,t,sk)|) \\ &\quad + \frac{Ck^4}{t(1+t)}\left[|\hat{v}(t)|(1+t^2)(|S_J(k,t,sk)| + |S_K(k,t,sk)|) \right. \\ &\quad \left. + |S_J(k,t,sk)|^2 + |S_K(k,t,sk)|^2\right]. \end{aligned} \quad (6.133)$$

For $2k \leq t \leq 1$, the potential and S_J, S_K are uniformly bounded. Hence

$$|R_3(k,t,sk)| \leq Ck^3t + \frac{Ck^4}{t}, \quad (6.134)$$

whose integral, divided by $\omega_{\text{bg}}(k)$, is bounded by $Ck^2 + Ck^3|\log k|$.

For $t \geq 1$, Lemma 6.6 and (6.132) give, uniformly in k, s ,

$$\int_1^{\infty} \frac{dt}{t^2} (|S_J(k,t,sk)| + |S_K(k,t,sk)|)^2 \leq C. \quad (6.135)$$

Moreover, Cauchy–Schwarz yields

$$\begin{aligned} &\int_1^{\infty} dt |\hat{v}(t)|(|S_J(k,t,sk)| + |S_K(k,t,sk)|) \\ &\leq \left(\int_1^{\infty} t^2 \hat{v}(t)^2 dt\right)^{\frac{1}{2}} \left(\int_1^{\infty} \frac{dt}{t^2} (|S_J(k,t,sk)| + |S_K(k,t,sk)|)^2\right)^{\frac{1}{2}} \leq C. \end{aligned} \quad (6.136)$$

Since $\frac{t^2}{\omega_{\text{bg}}(t)}$ and $(1+t^2)(t(1+t))^{-1}$ are bounded on this interval, its contribution is at most $\frac{C(k^3+k^4)}{\omega_{\text{bg}}(k)} \leq Ck^2$. Moreover, The density factor is uniformly bounded. Combining the two intervals proves the assertion. $\square$

Combining the four lemmas with (6.107), we obtain, in normalized units,

$$\delta(I_{\mathbf{k},3}) = -c_{\text{ir}}(k, 1, \hat{v}) + k(c_{\text{main}}(\hat{v}) + c_{\text{shift}}(\hat{v})) + o(k).$$

Since the contributions from $I_{\mathbf{k},1}$ and $I_{\mathbf{k},2}$ are, respectively, $o(k)$ and $-\frac{15k}{16\pi^2} + o(k)$, this convention gives

$$c_3(1, \hat{v}) = c_{\text{main}}(\hat{v}) + c_{\text{shift}}(\hat{v}) - \frac{15}{16\pi^2}.$$

which can be seen to coincide with the coefficient in (3.67)–(3.69) after restoring ν and $\hat{v}(0)$ as in Remark 6.1.

APPENDIX A. SCATTERING LENGTH

In this appedix we define the scattering length and derive the Born series, which allows to compute it, essentially following [30], with our notation and conventions. Note that our definition differs from the corresponding one in [30] by a factor of 2 in front of the potential.

We start from a Schrödinger operator $-\Delta + \kappa v(x)$ on $L^2(\mathbb{R}^3)$ with v sufficiently fast decaying and spherically symmetric. First we look for a zero-energy generalized

eigenfunction of this operator. We make an ansatz saying that this eigenfunction has the form $1 - w(x)$, where w is of the order κ , and has the asymptotic form

$$w(x) \simeq \frac{\mathbf{a}}{|x|}, \quad |x| \rightarrow \infty, \quad (\text{A.1})$$

where $\mathbf{a}$ is called the *scattering length*. Now the eigenvalue equation

$$(-\Delta + \kappa v(\mathbf{x}))(1 - w(\mathbf{x})) = 0 \quad (\text{A.2})$$

can be rewritten as

$$\kappa v = (1 + \kappa v(-\Delta)^{-1})(-\Delta w). \quad (\text{A.3})$$

Hence

$$-\Delta w = \sum_{n=0}^{\infty} (-1)^n \kappa^{n+1} (v(-\Delta)^{-1})^n v. \quad (\text{A.4})$$

In the momentum representation this can be rewritten as

$$\begin{aligned} -\widehat{\Delta w}(\mathbf{p}) &= \kappa \hat{v}(\mathbf{p}) \\ &+ \sum_{k=1}^{+\infty} (-1)^k \kappa^{k+1} \int \frac{d\mathbf{p}_1 \dots d\mathbf{p}_k}{((2\pi)^3)^k} \frac{\hat{v}(\mathbf{p} - \mathbf{p}_1) \hat{v}(\mathbf{p}_1 - \mathbf{p}_2) \dots \hat{v}(\mathbf{p}_{k-1} - \mathbf{p}_k) \hat{v}(\mathbf{p}_k)}{|\mathbf{p}_1|^2 \dots |\mathbf{p}_k|^2}, \end{aligned} \quad (\text{A.5})$$

Now

$$-\widehat{\Delta w}(\mathbf{0}) = - \int \Delta w(\mathbf{x}) d\mathbf{x} = - \lim_{R \rightarrow +\infty} \int_{|\mathbf{x}| < R} \Delta w(\mathbf{x}) d\mathbf{x} \quad (\text{A.6})$$

$$= \lim_{R \rightarrow +\infty} \int_{|\mathbf{x}|=R} \nabla w(\mathbf{x}) \cdot d\vec{\Omega}_R = \lim_{R \rightarrow +\infty} \int_{|\mathbf{x}|=R} \frac{\mathbf{a}}{R^2} d\Omega_R = 4\pi \mathbf{a}, \quad (\text{A.7})$$

where Ω_R is the usual measure on the sphere of radius R and $\vec{\Omega}_R$ is the same measure multiplied by the outward normal vector. Setting $\mathbf{p} = \mathbf{0}$ we obtain

$$\begin{aligned} 4\pi \mathbf{a} &= \kappa \hat{v}(\mathbf{0}) \\ &+ \sum_{k=1}^{+\infty} (-1)^k \kappa^{k+1} \int \frac{d\mathbf{p}_1 \dots d\mathbf{p}_k}{((2\pi)^3)^k} \frac{\hat{v}(\mathbf{p}_1) \hat{v}(\mathbf{p}_1 - \mathbf{p}_2) \dots \hat{v}(\mathbf{p}_{k-1} - \mathbf{p}_k) \hat{v}(\mathbf{p}_k)}{|\mathbf{p}_1|^2 \dots |\mathbf{p}_k|^2}, \end{aligned}$$

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(J. Dereziński) DEPT. OF MATH. METHODS IN PHYS., UNIVERSITY OF WARSAW, PASTEURA 5, 02-093 WARSZAWA, POLAND

Email address: Jan.Derezinski@fuw.edu.pl

(L. Pettinari) DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI TRENTO AND INFN-TIFPA AND INdAM, VIA SOMMARIVE 14, I-38123 POVO, ITALY

Email address: lorenzo.pettinari@unitn.it