

DAMPING OF PHONONS IN BOSE GAS AT LOW TEMPERATURES

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ABSTRACT. We consider a homogeneous Bose gas in a large cubic box with periodic boundary conditions interacting with a small potential with a positive Fourier transform. We compute the imaginary part of the phononic excitation spectrum in the lowest order of perturbation theory in the thermodynamic limit at low temperatures and low momentum. Our analysis is based on perturbation theory of the standard Liouvillean. We use two approaches: the first, motivated by the standard representation of operator algebras, examines resonances near zero; the second analyzes the 2-point correlation function in the energy-momentum space.

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1. INTRODUCTION

We consider a homogeneous Bose gas with a two-body potential $v(\mathbf{x})$ having a positive Fourier transform at positive density and a (low) temperature $\frac{1}{k_B\beta}$. We assume that the density is positive, and is fixed by a positive parameter ν , which can be interpreted as the chemical potential. We assume at least partial Bose-Einstein condensation of the quantum gas, which is believed to be appropriate for

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low temperatures. We use the Hamiltonian with the zeroth mode replaced by a c -number. We work in a large periodic box of size $L \rightarrow \infty$.

We treat the quadratic part of our Hamiltonian as a good approximation of the system. This quadratic part can be diagonalized with the help of a Bogoliubov transformation [9]. Therefore, we call it the *Bogoliubov Hamiltonian*. As is well-known, this yields the quasiparticle dispersion relation

$$\omega_{\text{bg}}(\mathbf{k}) = \sqrt{\frac{1}{4}|\mathbf{k}|^4 + \frac{\nu \hat{v}(\mathbf{k})}{\hat{v}(\mathbf{0})}|\mathbf{k}|^2}. \quad (1.1)$$

This relation for low momenta behaves as $\omega_{\text{bg}}(\mathbf{k}) = \sqrt{\nu}|\mathbf{k}|$, and the corresponding quasiparticles are called *phonons*.

In the Bogoliubov approximation the interacting Bose gas is approximately described as the free Bose gas with a peculiar dispersion relation. It is remarkable that this picture to a large degree is validated by experiments with ^4He [4] and alkali gases. Neutron scattering (for larger momenta) and sound propagation (for small momenta) in ^4He allow us to measure its quasiparticle dispersion relation, which turns out to be a rather well-defined curve similar to its theoretical prediction (1.1). Experiments show that this relation is remarkably sharp, even though we expect it to be broadened by interaction with higher order terms of the Hamiltonians.

This broadening is the main topic of our paper. It can be described by the imaginary part of the dispersion relation computed in perturbation theory. To define the perturbation expansion we replace the potential v by κv , which yields $\sqrt{\kappa}$ in front of the three body interaction and κ in front of the four-body interaction.

For positive temperatures it is convenient to use the Liouvillean instead of the Hamiltonian. This involves doubling the original Hilbert space, which now possesses two kinds of quasiparticles: “excitations over the thermal equilibrium”, and “holes in thermal equilibrium”, which we will call the *left*, resp. *right quasiparticles*.

The main result of our paper is a computation of the imaginary part of the dispersion relation in the lowest nontrivial order in κ . This amounts to the Fermi Golden Rule for the three-body term. We do this both at zero temperature, and at positive temperature, keeping the chemical potential ν and the temperature $\frac{1}{k_B\beta}$ fixed, and taking thermodynamic limit $L \rightarrow \infty$.

We show that the imaginary part of the quasiparticle dispersion relation in the order κ^1 is the sum of two terms:

$$-\gamma_{\text{B}}(\mathbf{k}, \beta, \nu) - \gamma_{\text{L}}(\mathbf{k}, \beta, \nu). \quad (1.2)$$

In the literature the term γ_{B} goes under the name of the *Beliaev damping*. The Beliaev damping is caused by a left quasiparticle decaying into two left quasiparticles. This process persists down to the zero temperature. This decay rate was first computed by Beliaev in a series of two remarkable papers [5, 6]. This result has been re-derived following different theoretical approaches in several other works, e.g. [26, 31, 32, 45, 47]. In particular, the present paper can be viewed as a continuation of the recently published detailed derivation of the Beliaev damping at zero temperature by Dereziński, Li and Napiórkowski [26]. The damping rate observed in experiments appears to be consistent with its theoretical predictions [21, 28].

The term γ_{L} is usually called the *Landau damping*. It was first computed by Hohenberg and Martin in [37] and later by other authors [32, 45, 53]. Also in this case, experimental results [20] seem to validate the theoretical predictions; see [45] for a comparison between experimental and theoretical results. This process is absent at zero temperature, as it involves an interaction between left and right quasiparticles. Indeed, it is caused by a left quasiparticle decaying into one left and one right quasiparticle.

Before describing the formulas for (1.2), let us introduce the auxiliary functions

$$s_{\mathbf{k}} := \text{sign}(\hat{v}(\mathbf{k})) \frac{\sqrt{\sqrt{\omega_{\text{bg}}(\mathbf{k})^2 + \nu^2 \frac{\hat{v}(\mathbf{k})^2}{\hat{v}(\mathbf{0})^2}} - \omega_{\text{bg}}(\mathbf{k})}}{\sqrt{2\omega_{\text{bg}}(\mathbf{k})}}, \quad (1.3)$$

$$c_{\mathbf{k}} := \frac{\sqrt{\sqrt{\omega_{\text{bg}}(\mathbf{k})^2 + \nu^2 \frac{\hat{v}(\mathbf{k})^2}{\hat{v}(\mathbf{0})^2}} + \omega_{\text{bg}}(\mathbf{k})}}{\sqrt{2\omega_{\text{bg}}(\mathbf{k})}}. \quad (1.4)$$

They are the coefficients of the Bogoliubov transformation diagonalizing the quadratic part of the Hamiltonian: note that $c_{\mathbf{k}}^2 - s_{\mathbf{k}}^2 = 1$. For three momenta $\mathbf{k}, \mathbf{p}, \mathbf{q}$, we set

$$j(\mathbf{k}; \mathbf{p}, \mathbf{q}) := \sqrt{\frac{\nu}{\hat{v}(\mathbf{0})}} \left[\hat{v}(\mathbf{k})(s_{\mathbf{k}} - c_{\mathbf{k}})(c_{\mathbf{p}}s_{\mathbf{q}} + c_{\mathbf{q}}s_{\mathbf{p}}) \right. \\ \left. + \hat{v}(\mathbf{p})(c_{\mathbf{p}} - s_{\mathbf{p}})(c_{\mathbf{k}}c_{\mathbf{q}} + s_{\mathbf{k}}s_{\mathbf{q}}) \right. \\ \left. + \hat{v}(\mathbf{q})(c_{\mathbf{q}} - s_{\mathbf{q}})(c_{\mathbf{p}}c_{\mathbf{k}} + s_{\mathbf{p}}s_{\mathbf{k}}) \right], \quad (1.5)$$

and

$$\kappa(\mathbf{k}, \mathbf{p}, \mathbf{q}) := \sqrt{\frac{\nu}{\hat{v}(\mathbf{0})}} \left[\hat{v}(\mathbf{k})(s_{\mathbf{k}} - c_{\mathbf{k}})(c_{\mathbf{p}}s_{\mathbf{q}} + c_{\mathbf{q}}s_{\mathbf{p}}) \right. \\ \left. + \hat{v}(\mathbf{p})(s_{\mathbf{p}} - c_{\mathbf{p}})(c_{\mathbf{k}}s_{\mathbf{q}} + s_{\mathbf{k}}c_{\mathbf{q}}) \right. \\ \left. + \hat{v}(\mathbf{q})(s_{\mathbf{q}} - c_{\mathbf{q}})(c_{\mathbf{p}}s_{\mathbf{k}} + s_{\mathbf{p}}c_{\mathbf{k}}) \right], \quad (1.6)$$

The functions $j(\mathbf{k}; \mathbf{p}, \mathbf{q}) = j(\mathbf{k}; \mathbf{q}, \mathbf{p})$ and $\kappa(\mathbf{k}, \mathbf{p}, \mathbf{q}) = \kappa(\mathbf{p}, \mathbf{k}, \mathbf{q}) = \kappa(\mathbf{p}, \mathbf{q}, \mathbf{k})$ describe the amplitude of a three-quasiparticle vertex. We prove that

$$\gamma_{\text{B}} = \frac{\pi}{2} \int \frac{d\mathbf{p}}{(2\pi)^3} j(\mathbf{k}; \mathbf{p}, \mathbf{p} - \mathbf{k})^2 \delta(\omega_{\text{bg}}(\mathbf{k}) - \omega_{\text{bg}}(\mathbf{p}) - \omega_{\text{bg}}(\mathbf{k} - \mathbf{p})) \\ \times \frac{(1 - e^{\frac{\beta}{2}(\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{k} - \mathbf{p}))})^2}{(e^{\beta\omega_{\text{bg}}(\mathbf{k})} - 1)(e^{\beta\omega_{\text{bg}}(\mathbf{p})} - 1)(e^{\beta\omega_{\text{bg}}(\mathbf{k} - \mathbf{p})} - 1)}, \quad (1.7)$$

$$\gamma_{\text{L}} = \pi \int \frac{d\mathbf{p}}{(2\pi)^3} j(\mathbf{p} - \mathbf{k}; \mathbf{k}, \mathbf{p})^2 \delta(\omega_{\text{bg}}(\mathbf{k} - \mathbf{p}) - \omega_{\text{bg}}(\mathbf{p}) - \omega_{\text{bg}}(\mathbf{k})) \\ \times \frac{(e^{\frac{\beta}{2}(\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{k} - \mathbf{p}))} - e^{\frac{\beta}{2}\omega_{\text{bg}}(\mathbf{p})})^2}{(e^{\beta\omega_{\text{bg}}(\mathbf{k})} - 1)(e^{\beta\omega_{\text{bg}}(\mathbf{p})} - 1)(e^{\beta\omega_{\text{bg}}(\mathbf{k} - \mathbf{p})} - 1)}. \quad (1.8)$$

The function $\kappa(\mathbf{k}, \mathbf{p}, \mathbf{q})$ will be relevant in the computation of the quasi-particle two-point function.

We also analyze the behavior of γ_{B} for small momenta, and of γ_{L} for small momenta and temperatures. To understand these results one should first remark that a natural dimensionless measure of the momentum is $\frac{|\mathbf{k}|}{\sqrt{\nu}}$, and of temperature is $\frac{1}{\beta\nu}$. Thus the ratio of momentum to temperature is described by $\beta|\mathbf{k}|\sqrt{\nu}$.

In both theorems we make some additional assumptions on the potential, which essentially say that the Fourier transform of v is not too negative and is sufficiently differentiable. Some additional hypotheses for the characterization of the shape of the Bogoliubov dispersion relation will be needed. They will be explained in detail in Subsect. 3.1.

The following theorem describes the Beliaev damping rate for small momenta.

Theorem 1.1. *The following estimates hold:*

(1) For small momenta and temperature/momentum ratios we have

$$\gamma_B(\mathbf{k}, \beta, \nu) = \frac{3\hat{v}(\mathbf{0})\nu^{3/2}}{640\pi} \frac{|\mathbf{k}|^5}{\nu^{5/2}} \left(1 + O\left(\frac{1}{(\beta\sqrt{\nu}|\mathbf{k}|)^3}\right) + O\left(\frac{|\mathbf{k}|^2}{\nu}\right) \right) \\ \text{as } \frac{|\mathbf{k}|}{\sqrt{\nu}}, \frac{1}{\beta\sqrt{\nu}|\mathbf{k}|} \rightarrow 0. \quad (1.9)$$

(2) For small momenta and momentum/temperature ratios we have

$$\gamma_B(\mathbf{k}; \beta, \nu) = \frac{3\hat{v}(\mathbf{0})\nu^{3/2}}{128\pi} \frac{|\mathbf{k}|^4}{\nu^2} \frac{1}{\beta\nu} \left(1 + O(\beta\sqrt{\nu}|\mathbf{k}|) + O\left(\frac{|\mathbf{k}|^2}{\nu}\right) \right) \\ \text{as } \frac{|\mathbf{k}|}{\sqrt{\nu}}, \beta\sqrt{\nu}|\mathbf{k}| \rightarrow 0. \quad (1.10)$$

Next we describe the asymptotics of the Landau damping rate for small momenta and temperatures.

Theorem 1.2. *The following estimates hold:*

(1) For small momenta, temperatures and the momentum/temperature ratios we have

$$\gamma_L(\mathbf{k}; \beta, \nu) = \frac{3\pi^3\hat{v}(\mathbf{0})\nu^{3/2}}{40} \frac{|\mathbf{k}|}{\sqrt{\nu}} \frac{1}{(\beta\nu)^4} \left(1 + O(\beta\sqrt{\nu}|\mathbf{k}|) + O\left(\frac{1}{(\beta\nu)^2}\right) \right) \\ \text{as } \frac{|\mathbf{k}|}{\sqrt{\nu}}, \frac{1}{\beta\nu}, \beta\sqrt{\nu}|\mathbf{k}| \rightarrow 0. \quad (1.11)$$

(2) For small momenta, temperatures and the temperature/momentum ratios we have

$$\gamma_L(\mathbf{k}; \beta, \nu) = \frac{9\zeta(3)\hat{v}(\mathbf{0})\nu^{3/2}}{16\pi} \frac{1}{(\beta\nu)^3} \frac{|\mathbf{k}|^2}{\nu} \left(1 + O\left(\frac{1}{\beta\sqrt{\nu}|\mathbf{k}|}\right) + O\left(\frac{|\mathbf{k}|^2}{\nu}\right) \right) \\ \text{as } \frac{|\mathbf{k}|}{\sqrt{\nu}}, \frac{1}{\beta\nu}, \frac{1}{\beta\sqrt{\nu}|\mathbf{k}|} \rightarrow 0. \quad (1.12)$$

For $\beta\sqrt{\nu}|\mathbf{k}| \rightarrow 0$, the function $\gamma_L(\mathbf{k}; \beta, \nu)$ approaches the known low-temperature value of the Landau damping rate as originally derived by Hohenberg and Martin in [37]. To the best of our knowledge, the temperature dependent correction (1.10) for $\gamma_B(\mathbf{k}; \beta, \nu)$ as $\beta\sqrt{\nu}|\mathbf{k}| \rightarrow 0$ is first obtained here. In this regime, the ratio between the Landau and the Beliaev damping is of order

$$\frac{\gamma_B(\mathbf{k}; \beta, \nu)}{\gamma_L(\mathbf{k}; \beta, \nu)} = O((\beta\sqrt{\nu}|\mathbf{k}|)^3) \quad \text{as } \beta\sqrt{\nu}|\mathbf{k}| \rightarrow 0, \quad (1.13)$$

that is, the dominant effect is given by the Landau damping.

Conversely, in the limit $\frac{1}{\beta\sqrt{\nu}|\mathbf{k}|} \rightarrow 0$, $\gamma_B(\mathbf{k}; \beta, \nu)$ reproduces the standard expression originally computed in [6]. In the same asymptotic regime Equation (1.12) for the Landau damping appears to be a novel contribution of the present paper. The ratio of the two rates is of order

$$\frac{\gamma_L(\mathbf{k}; \beta, \nu)}{\gamma_B(\mathbf{k}; \beta, \nu)} = O\left(\frac{1}{(\beta\sqrt{\nu}|\mathbf{k}|)^3}\right) \quad \text{as } \frac{1}{\beta\sqrt{\nu}|\mathbf{k}|} \rightarrow 0, \quad (1.14)$$

and the dominant contribution is the Beliaev damping one.

It was further pointed out in [45] that there exist experimental setups where the approximation $\frac{1}{\beta\nu} \rightarrow 0$ is not well justified. In that case, one should employ the full integral expressions (1.7) and (1.8). We give a brief analysis of the *high temperature regime* $\beta\nu \rightarrow 0$ in Proposition 3.13, although its physical relevance is somewhat dubious.

So far we have tried to describe the physical meaning of our results without discussing the formalism used to derive them. Actually, we will use two distinct

formalisms, both yielding the formulas (1.7) and (1.8) for the damping rates. Both of them, described in the first part of our paper, are based on the Hamiltonian with a c -number condensate, and the corresponding Liouvillean. Let us call them

- (1) the *standard representation approach*,
- (2) the *Green function approach*.

The first formalism is directly inspired by the modular theory of W^* -algebras [11, 12, 23]. We first consider the Bogoliubov Liouvillean, that is the Liouvillean obtained from the purely quadratic part of the Hamiltonian. It is a quadratic bosonic operator, and therefore its W^* -algebraic theory is well understood. One can consider one-quasiparticle states and their corresponding standard vector representatives, which belong to the so-called standard positive cone. They are embedded in continuous spectrum, and under the influence of the perturbation in thermodynamic limit they develop an imaginary part of their energy. These vectors are obtained by acting on the KMS vector with one left and one right quasiparticle creation operator.

Similar computations, pioneered by Jakšić-Pillet, were applied in the literature to spin-boson [39] and Pauli-Fierz-type Hamiltonians [25], where they involved non-trivial (type III) W^* -algebraic systems directly in thermodynamic limit. In these applications an infinitely extended system interacted with a small one, and there was no translation invariance. In our case we consider a sequence of finite volume systems whose size goes to infinity. Our W^* -algebras are trivial from the point of view of the abstract theory (they are of type I). Nevertheless, in our opinion the methods developed in the operator algebras approach yield useful guiding principles for physical systems.

The second approach involves time-ordered two-point correlation functions of the fields in the energy-momentum space. It was originally inspired by the formalism of quantum electrodynamics, and was used by Beliaev in [5, 6]. The correlation functions often go under the name of Green functions, since by the Lehmann representation they are expectation values of the resolvent of the Liouvillian. This approach belongs to standard tools of many body quantum physics [36].

The pure Bogoliubov Liouvillean has a singular shell along the dispersion relation of quasiparticles, both in the upper halfspace and symmetrically in the lower halfspace. These singular shells are produced by acting with one quasiparticle creation operator onto the Bogoliubov KMS vector. Under the perturbation, the dispersion relation should develop an imaginary part, so that the shells become broadened. This broadening is visible in 2-point functions.

The two-point functions are convenient theoretically, however they seem to be difficult to access empirically. Typical experiments measure the so-called van Hove form factors, expressible in terms of density-density correlation functions. They are special cases of 4-point correlation functions. One expects that the energy-momentum picture of density-density correlation functions is similar to that of 2-point functions [31, 34].

Anyway, both the standard representation approach and the Green function approach are consistent with one another, at least to the lowest nontrivial order in perturbation theory, and yield the same damping rate given in (1.2), (1.7), and (1.8). They both rely on a *weak-coupling approximation*, where the small parameter governing the expansion of the energy corrections is the Fourier transform of the interaction potential. As we mentioned above, this is made explicit through the introduction of a coupling constant κ in front of the potential, while keeping the chemical potential ν fixed.

In most physics literature when considering phonon damping one uses the so-called low-density approximation, where the potential is not necessarily small, but

very short range [5, 6, 31, 35, 46, 57]. The small parameter is then the scattering length $\mathfrak{a}$. The low-density approximation is usually considered to be more physically meaningful, relevant especially for Bose condensates of alkali elements. We prefer to use the weak coupling limit, since mathematically it is simpler and cleaner.

One can also remark that the most famous Bose condensate, that is ^4He below the critical temperature, is neither weakly interacting, nor dilute. Its quasiparticle dispersion relation has a complicated shape with *rotons*, *maxons* and the *Pitaevskii plateau*, qualitatively similar to the picture obtained in the weak coupling limit.

In fact, in the dilute approximation, instead of (1.1), the dispersion relation reads

$$\omega_{\text{bg}}(\mathbf{k}) = \sqrt{\frac{1}{4}|\mathbf{k}|^4 + n_0\mathfrak{a}|\mathbf{k}|^2}, \quad (1.15)$$

where $\mathfrak{a}$ is the scattering length and n_0 is the density of the condensate. Now, (1.15) is strictly monotonic, without maxons or rotons.

When investigating the quasiparticle spectrum of Bose gas from “first principles” one assumes that the Hamiltonian is in a large box with periodic boundary conditions, and then goes to thermodynamic limit. One needs to arrange this Hamiltonian in such a way that one can extract the quadratic part, which yields the Bogoliubov approximation. This can be done in at least two settings:

- (1) the canonical approach,
- (2) the grand-canonical approach,

In the canonical approach one can start from the N -body Hamiltonian with a fixed number of particles. Following the idea of Arnowitt-Girardeau [3], one can transform the N -body space into a Fock space with a variable number of particles by treating the zero mode in a special way. This transformation was used e.g. by Dereziński-Napiórkowski [24], and essentially also by Lewin-Nam-Serfaty-Solovej [43]. One keeps the density of particles $\frac{N}{L^d}$ fixed. If we retain in the Hamiltonian only the terms which have “nice thermodynamic limits”, then we obtain the Hamiltonian with a c -number substitution.

The second approach follows Hugenholtz-Pines [38], and is described e.g. in [26]. It starts from the grand-canonical Hamiltonian, which allows for a variable number of particles from the very beginning and is obtained by subtracting νN from the Hamiltonian. The chemical potential ν is kept fixed in thermodynamic limit. If at the end we compress our Fock space away from the zeroth mode, we also obtain the Hamiltonian with a c -number substitution, the same as before.

We should mention yet another approach, which is implemented for example in [41, 46, 47], where one makes use of a non-self-adjoint Hamiltonian. We will not explore this direction in the present paper.

One of the important elements of our analysis is keeping fixed the parameter ν . In a more precise analysis we actually encounter several possibilities for the interpretation of the parameter ν .

- (1) In the grand-canonical approach ν is the true chemical potential (then we are forced to use the formalism with a variable number of particles).
- (2) ν can be the density times $\kappa \int v(\mathbf{x})d\mathbf{x}$. This is natural if we use the canonical formalism, with a fixed number of particles.
- (3) ν can be the density of the condensate (the zero mode) times $\kappa \int v(\mathbf{x})d\mathbf{x}$.

Anyway, these three interpretations differ (at least formally) by a quantity of the order κ , and therefore they do not affect the imaginary part of the dispersion relation in the lowest order.

As discussed above, within the weak-coupling approximation, our results reproduce the known decay rates (1.9) and (1.11) in the corresponding temperature and momentum regimes. In principle, the Green-function approach should determine

both the imaginary and the real parts of the quasiparticle dispersion relation. The approximations employed in the present work, however, are not sufficiently accurate to compute the real part correctly. Even worse, the resulting correction to the energy of a quasiparticle with momentum $\mathbf{k}$ exhibits an unphysical infrared divergence of order $\frac{1}{|\mathbf{k}|}$. We now explain the origin of this failure. Throughout this paper, we assume that the Bose gas is described by a Hamiltonian with a c -number condensate. Formally, this Hamiltonian is obtained from the usual first-principles Hamiltonian of the Bose gas by replacing the creation and annihilation operators of the zero mode by $\sqrt{\nu} \in \mathbb{R}$ and dropping an additive constant. The resulting Hamiltonian takes the form

$$H^L = H_{\text{bg}}^L + \sqrt{\kappa} H_3^L + \kappa H_4^L, \quad (1.16)$$

and the corresponding Liouvillean has the analogous decomposition

$$L^L = L_{\text{bg}}^L + \sqrt{\kappa} L_3^L + \kappa L_4^L. \quad (1.17)$$

Here, the superscript L denotes the size of the box, and the limit $L \rightarrow \infty$ is taken in our computations. In perturbation theory for the quasiparticle spectrum, the contribution of order $\sqrt{\kappa}$ vanishes. The leading correction is therefore of order κ and has the form

$$(\Psi_0 | L_4^L \Psi_0) + (L_3^L \Psi_0 | (L_{\text{bg}}^L - i\epsilon)^{-1} L_3^L \Psi_0), \quad (1.18)$$

where Ψ_0 denotes an appropriate unperturbed vector. Thus, (1.18) is the sum of a Hellmann-Feynman term involving L_4^L and a Fermi Golden Rule term involving L_3^L . The imaginary part of the correction comes entirely from the Fermi Golden Rule term and therefore depends only on L_3^L . As shown in [27], the c -number substitution produces the correct cubic contribution L_3^L . This explains why the present approach correctly captures the damping rates. By contrast, the real part of the dispersion relation receives contributions from both terms in (1.18). The quartic term L_4^L obtained through the naive c -number substitution misses non-negligible contributions generated by the commutation relations of the zero-mode creation and annihilation operators. This issue was already observed by Hugenholtz and Pines [38] in the grand-canonical formalism.

At zero temperature in the dilute approximation the real shift of the dispersion relation was computed already by Beliaev [6]. In the weak coupling regime this real shift is the subject of our second paper on this topic [27]. We discuss both the canonical and grand-canonical settings at zero temperature. Careful analysis leads to additional terms in the Hamiltonian, absent the naive Hamiltonian with a c -number substitution. Some of them are of order κ^1 , others are of higher order and/or (formally) vanish in thermodynamic limit. We discuss them in the separate paper [27]. They are necessary when we compute the real part of the corrections to the quasiparticle spectrum. The real shift requires an analysis much finer than the naive c -number substitution used in this paper.

Let us conclude with some additional remarks on the literature. In recent years, there have been numerous rigorous justifications of Bogoliubov's c -number substitution in various physical regimes. In the mean-field setting, this was first established by Seiringer [55]; see also [43, 24, 33, 50] for related developments. Corresponding results in the Gross-Pitaevskii regime were subsequently obtained in [7, 14, 51], and extensions beyond this regime appear in [13].

Operator-algebraic methods have also been employed to construct C^* -dynamical systems describing interacting Bose gases [15, 16]. A different strategy, based on techniques from quantum field theory, was developed by Galanda and Pinamonti in [29, 30], where equilibrium states for an interacting Bose gas are constructed via a suitable resummation of perturbative expansions.

Further advances concerning the bottom of the excitation spectrum beyond Bogoliubov theory have recently been achieved in the Gross–Pitaevskii regime [18]. In the mean-field regime, analogous analyses were carried out in [10, 48]. Finally, the energy–momentum spectrum of Bose gases was first rigorously investigated in [24, 26].

Apart from the introduction, the paper consists of two main sections. In Section 2 we discuss the general theory of quasiparticle spectrum at zero and low positive temperatures. In particular, we introduce the two approaches to define the phonon decay rate. In Section 3, we will formulate precisely and prove Theorems 1.1 and 1.2 stated in the introduction about the asymptotics of the phonon damping for low momenta. We will consider the (physical) dimension 3 and we will impose additional technical assumptions on the potential.

2. QUASIPARTICLE SPECTRUM OF BOSE GAS

2.1. The “first-principles Hamiltonian”. Consider a real function $\mathbb{R}^d \ni \mathbf{x} \mapsto v(\mathbf{x})$, describing the two-body potential of a quantum gas. Its Fourier transform will be defined by

$$\hat{v}(\mathbf{p}) := \int d\mathbf{x} \, v(\mathbf{x}) e^{-i\mathbf{p}\mathbf{x}}. \quad (2.1)$$

A large part of our preliminary analysis will be general and will not depend on the dimension nor on the regularity assumptions on the potential. Here we just mention the most basic assumptions, which we will need to describe the general theory.

We assume that $\nu \in]0, \nu_0]$, (for some $\nu_0 > 0$) which has the interpretation of the effective chemical potential. Moreover, we will always assume

$$v \in L^1(\mathbb{R}^d, d\mathbf{x}), \, v(\mathbf{x}) \in \mathbb{R}; \quad (2.2a)$$

$$\hat{v} \in L^2(\mathbb{R}^d, d\mathbf{k}); \quad (2.2b)$$

$$\hat{v}(\mathbf{0}) > 0, \, \hat{v}(\mathbf{k}) > -\hat{v}(\mathbf{0}) \frac{|\mathbf{k}|^2}{4\nu}; \quad \nu \in]0, \nu_0]; \quad (2.2c)$$

$$v(\mathbf{x}) = v(-\mathbf{x}), \quad \text{which implies } \hat{v}(\mathbf{k}) = \hat{v}(-\mathbf{k}). \quad (2.2d)$$

The first hypotheses, (2.2a) guarantees that the Fourier transform of v , as in (2.1), is a continuous function and the Hamiltonian is self-adjoint. The second one (2.2b) provides a convenient natural cut-off on the high momenta region. The third one, (2.2c), is needed to make sense of the Bogoliubov transformation, *cf.* Equation (2.18), where the linear combination of $b_{\mathbf{k}}, \, b_{\mathbf{k}}^*$ can only be defined for $\frac{1}{2}|\mathbf{k}|^2 + \nu \frac{\hat{v}(\mathbf{k})}{\hat{v}(\mathbf{0})} > 0$. The last condition, (2.2d), can be always imposed if we deal with identical particles (Bose or Fermi).

In our detailed computations, described in Section 3, we fix $d = 3$, and we will assume some additional restrictions on the class of interaction potential v we consider. They will be described in Subsect. 3.1.

We consider Bose gas with potential v in large but finite volume with periodic boundary conditions. Following the standard approach, we replace the infinite space $\mathbb{R}^d$ by the torus $\Lambda_L =]-L/2, L/2]^d$. Let $\Xi_L := \frac{2\pi}{L}\mathbb{Z}^d$ be the momentum lattice corresponding to Λ_L . In the momentum representation and the 2nd quantized formalism, the Hamiltonian, number operator and total momentum are operators

on the Fock space $\Gamma_s(l^2(\Xi_L))$ given by

$$H^L = \sum_{\mathbf{p}} \frac{1}{2} \mathbf{p}^2 a_{\mathbf{p}}^* a_{\mathbf{p}} \quad (2.3)$$

$$+ \frac{1}{2L^d} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{k}} \hat{v}(\mathbf{k}) a_{\mathbf{p}+\mathbf{k}}^* a_{\mathbf{q}-\mathbf{k}}^* a_{\mathbf{p}} a_{\mathbf{q}},$$

$$N^L = \sum_{\mathbf{p}} a_{\mathbf{p}}^* a_{\mathbf{p}}, \quad (2.4)$$

$$\mathbf{P}^L = \sum_{\mathbf{p}} \mathbf{p} a_{\mathbf{p}}^* a_{\mathbf{p}}, \quad (2.5)$$

where all momenta are summed over Ξ_L . We will sometimes call (2.3) the “first-principles Hamiltonian in finite volume”.

2.2. Hamiltonian with a c -number condensate. In this and in the following sections we will always implicitly assume the validity of (2.2a), (2.2b), (2.2c) and (2.2d). Let us introduce a parameter $\nu > 0$. Set $\Xi_L^> := \Xi_L \setminus \{\mathbf{0}\}$. Our basic Hamiltonian will not be (2.3), but the *Hamiltonian of the Bose gas with a c -number condensate*. It acts on $\Gamma_s(l^2(\Xi_L^>))$ and is defined by

$$H^L = H_{\text{bg}}^L + H_3^L + H_4^L, \quad (2.6)$$

$$H_{\text{bg}}^L := \sum_{\mathbf{p} \in \Xi_L^>} \left(\frac{\mathbf{p}^2}{2} + \frac{\nu \hat{v}(\mathbf{p})}{\hat{v}(\mathbf{0})} \right) a_{\mathbf{p}}^* a_{\mathbf{p}} + \sum_{\mathbf{p} \in \Xi_L^>} \left(\frac{\nu \hat{v}(\mathbf{p})}{2\hat{v}(\mathbf{0})} a_{\mathbf{p}} a_{-\mathbf{p}} + \text{h. c.} \right), \quad (2.7)$$

$$H_3^L := \frac{1}{L^{d/2}} \sum_{\mathbf{k}, \mathbf{p}, \mathbf{p}+\mathbf{k} \in \Xi_L^>} \frac{\sqrt{\nu} \hat{v}(\mathbf{k})}{\sqrt{\hat{v}(\mathbf{0})}} \left(a_{\mathbf{p}+\mathbf{k}}^* a_{\mathbf{k}} a_{\mathbf{p}} + \text{h. c.} \right), \quad (2.8)$$

$$H_4^L := \frac{1}{2L^d} \sum_{\substack{\mathbf{p}, \mathbf{q}, \\ \mathbf{p}+\mathbf{k}, \mathbf{q}-\mathbf{k} \in \Xi_L^>}} \hat{v}(\mathbf{k}) a_{\mathbf{p}+\mathbf{k}}^* a_{\mathbf{q}-\mathbf{k}}^* a_{\mathbf{p}} a_{\mathbf{q}}. \quad (2.9)$$

The Hamiltonian H^L can be obtained from H^L by replacing $\frac{\sqrt{\hat{v}(\mathbf{0})}}{L^{\frac{d}{2}}} a_0, \frac{\sqrt{\hat{v}(\mathbf{0})}}{L^{\frac{d}{2}}} a_0^*$ with $\sqrt{\nu}$, and dropping a constant.

We will usually replace ν with $\kappa \nu$, assuming that κ is small. Thus parameter ν can be interpreted as $\kappa \hat{v}(\mathbf{0}) \frac{N_0}{L^d}$. It will be kept fixed and independent of κ , so that H_{bg}^L does not depend on κ . Note that H_3^L , resp. H_4^L consists of terms of the order $\sqrt{\kappa}$, resp. κ . Thus (2.6) should be replaced by

$$H^L = H_{\text{bg}}^L + \sqrt{\kappa} H_3^L + \kappa H_4^L. \quad (2.10)$$

2.3. Thermodynamic limit. We would like to study Bose gas in thermodynamic limit $L \rightarrow \infty$, keeping $\nu > 0$ and $\mathbb{R}^d \ni \mathbf{k} \mapsto \hat{v}(\mathbf{k})$ fixed. $\Xi_L^>$ formally converges to $\mathbb{R}^d$. One should replace $\frac{1}{L^d} \sum_{\mathbf{k} \in \Xi_L^>}$ with $\frac{1}{(2\pi)^d} \int d\mathbf{k}$, and the operators $a_{\mathbf{k}}, a_{\mathbf{k}}^*$ for

$\mathbf{k} \in \Xi_L^>$ with $\frac{(2\pi)^{\frac{d}{2}}}{L^{\frac{d}{2}}} a_{\mathbf{k}}, \frac{(2\pi)^{\frac{d}{2}}}{L^{\frac{d}{2}}} a_{\mathbf{k}}^*$ for $\mathbf{k} \in \mathbb{R}^d$.

Thus H_{bg}^L , H_3^L , H_4^L and $\mathbf{P}^L$ formally converge to operators on $\Gamma_s(L^2(\mathbb{R}^d))$

$$H_{\text{bg}} := \int d\mathbf{p} \left(\frac{\mathbf{p}^2}{2} + \frac{\nu \hat{v}(\mathbf{p})}{\hat{v}(\mathbf{0})} \right) a_{\mathbf{p}}^* a_{\mathbf{p}} + \int d\mathbf{p} \left(\frac{\nu \hat{v}(\mathbf{p})}{2\hat{v}(\mathbf{0})} a_{\mathbf{p}} a_{-\mathbf{p}} + \text{h. c.} \right), \quad (2.11)$$

$$H_3 := \frac{1}{(2\pi)^{\frac{d}{2}}} \int d\mathbf{p} d\mathbf{k} \frac{\sqrt{\nu} \hat{v}(\mathbf{k})}{\sqrt{\hat{v}(\mathbf{0})}} \left(a_{\mathbf{p}+\mathbf{k}}^* a_{\mathbf{k}} a_{\mathbf{p}} + \text{h. c.} \right), \quad (2.12)$$

$$H_4 := \frac{1}{2(2\pi)^d} \int d\mathbf{p} d\mathbf{k} d\mathbf{q} \hat{v}(\mathbf{k}) a_{\mathbf{p}+\mathbf{k}}^* a_{\mathbf{q}-\mathbf{k}}^* a_{\mathbf{q}} a_{\mathbf{p}}, \quad (2.13)$$

$$\mathbf{P} := \int d\mathbf{p} \mathbf{p} a_{\mathbf{p}}^* a_{\mathbf{p}}. \quad (2.14)$$

Now

$$H := H_{\text{bg}} + \sqrt{\kappa} H_3 + \kappa H_4 \quad (2.15)$$

is the Hamiltonian of the Bose gas in infinite volume with a c -number condensate given by ν .

Note that we can also perform thermodynamic limit in the “first principles Hamiltonian” H^L of (2.3), and then we obtain H of (2.15) with $\nu = 0$.

In the literature the Hamiltonians H and H^L often appear in descriptions of Bose gas at positive density for large L . One should admit that the whole procedure of the c -number substitution is problematic. In particular, we do not know whether H in thermodynamic limit is well-defined as a self-adjoint operator. In any case, we prefer to ask questions about the limits as $L \rightarrow \infty$ of finite volume quantities involving H^L .

2.4. Bogoliubov method. The Hamiltonian H_{bg}^L , and also H_{bg} , will be called the *Bogoliubov Hamiltonian* and it will be treated as the main part of the full Hamiltonian. It is purely quadratic, therefore it can be diagonalized using an appropriate *Bogoliubov transformation*.

In fact, consider $\theta = (\theta_{\mathbf{k}})_{\mathbf{k} \in \Xi_L^>}$, a sequence of real numbers. Set

$$c_{\mathbf{k}} := \cosh \theta_{\mathbf{k}}, \quad s_{\mathbf{k}} := -\sinh \theta_{\mathbf{k}}. \quad (2.16)$$

For $\mathbf{k} \in \Xi_L^>$ we make the substitution

$$a_{\mathbf{k}}^* = c_{\mathbf{k}} b_{\mathbf{k}}^* - s_{\mathbf{k}} b_{-\mathbf{k}}, \quad a_{\mathbf{k}} = c_{\mathbf{k}} b_{\mathbf{k}} - s_{\mathbf{k}} b_{-\mathbf{k}}^*, \quad (2.17)$$

Note that we have $s_{\mathbf{k}} = s_{-\mathbf{k}}$ and $c_{\mathbf{k}} = c_{-\mathbf{k}} = \sqrt{1 + s_{\mathbf{k}}^2}$.

We choose the Bogoliubov rotation that kills double creators and annihilators, which amounts to choosing $s_{\mathbf{k}}$ and $c_{\mathbf{k}}$ as

$$c_{\mathbf{k}} = \frac{1}{\sqrt{2}} \left(\left(1 - \left(\frac{\hat{v}(\mathbf{k}) \frac{\nu}{\hat{v}(\mathbf{0})}}{\frac{1}{2}\mathbf{k}^2 + \hat{v}(\mathbf{k}) \frac{\nu}{\hat{v}(\mathbf{0})}} \right)^2 \right)^{-1/2} + 1 \right)^{1/2}. \quad (2.18)$$

$$s_{\mathbf{k}} = \frac{\nu \hat{v}(\mathbf{k})}{2\hat{v}(\mathbf{0})\omega_{\text{bg}}(\mathbf{k})c_{\mathbf{k}}} \quad (2.19)$$

Note that (2.18) and (2.19) are well defined on account of Assumption (2.2c). We obtain

$$H_{\text{bg}}^L = E_{\text{bg}}^L + \sum_{\mathbf{k} \in \Xi_L^>} \omega_{\text{bg}}(\mathbf{k}) b_{\mathbf{k}}^* b_{\mathbf{k}}, \quad (2.20)$$

$$\mathbf{P}^L = \sum_{\mathbf{k} \in \Xi_L^>} \mathbf{k} b_{\mathbf{k}}^* b_{\mathbf{k}}, \quad (2.21)$$

where the quasiparticle excitation spectrum is defined in Equation (1.1) and the ground state energy is

$$E_{\text{bg}}^L = - \sum_{\mathbf{k} \in \Xi_L^>} \frac{1}{2} \left(\frac{1}{2} \mathbf{k}^2 + \frac{\nu \hat{v}(\mathbf{k})}{\hat{v}(\mathbf{0})} - \omega_{\text{bg}}(\mathbf{k}) \right). \quad (2.22)$$

Introduce the unitary operator

$$U_\theta := \prod_{\mathbf{k} \in \Xi_L^>} e^{-\frac{1}{2} \theta_{\mathbf{k}} a_{\mathbf{k}}^* a_{-\mathbf{k}}^* + \frac{1}{2} \theta_{\mathbf{k}} a_{\mathbf{k}} a_{-\mathbf{k}}}. \quad (2.23)$$

Note that

$$U_\theta^* a_{\mathbf{k}} U_\theta = b_{\mathbf{k}}, \quad U_\theta^* a_{\mathbf{k}}^* U_\theta = b_{\mathbf{k}}^*.$$

We can rewrite (2.20) as

$$H_{\text{bg}}^L = E_{\text{bg}}^L + U_\theta^* \sum_{\mathbf{k} \in \Xi_L^>} \omega_{\text{bg}}(\mathbf{k}) a_{\mathbf{k}}^* a_{\mathbf{k}} U_\theta, \quad (2.24)$$

and the ground state of the Bogoliubov Hamiltonian is

$$\Omega_{\text{bg}} := U_\theta^* \Omega^> = \prod_{\mathbf{p} \in \Xi_L^>} \frac{1}{\cosh^{\frac{1}{2}} \theta_{\mathbf{p}}} e^{\frac{1}{2} \tanh \theta_{\mathbf{p}} a_{\mathbf{p}}^* a_{-\mathbf{p}}^*} \Omega^>, \quad (2.25)$$

where $\Omega^>$ is the vacuum in $\Gamma_s(l^2(\Xi_L^>))$.

Note that $\omega_{\text{bg}}(\mathbf{k})$ of (1.1) is well defined for all values $\mathbf{k} \in \mathbb{R}^d$, even though it is restricted to $\mathbf{k} \in \Xi_L^>$ in (2.20) and (2.24). Therefore, in thermodynamic limit the Bogoliubov Hamiltonian can be rewritten as

$$H_{\text{bg}} + \int d\mathbf{k} \omega_{\text{bg}}(\mathbf{k}) b_{\mathbf{k}}^* b_{\mathbf{k}}, \quad (2.26)$$

$$\mathbf{P} = \int d\mathbf{k} \mathbf{k} b_{\mathbf{k}}^* b_{\mathbf{k}}, \quad (2.27)$$

after subtracting from H_{bg} the infinite ground state energy, whose density is given by

$$\lim_{L \rightarrow +\infty} \frac{E_{\text{bg}}^L}{L^d} = - \frac{1}{2(2\pi)^d} \int d\mathbf{k} \left(\frac{1}{2} \mathbf{k}^2 + \frac{\nu \hat{v}(\mathbf{k})}{\hat{v}(\mathbf{0})} - \omega_{\text{bg}}(\mathbf{k}) \right). \quad (2.28)$$

The joint energy-momentum spectrum covers a part of the upper halfspace of $\mathbb{R}^{1+d}$. If the volume is finite, it is discrete and

$$\sigma(H_{\text{bg}}^L - E_{\text{bg}}^L, \mathbf{P}^L) \subset [0, \infty[\times \frac{2\pi}{L} \mathbb{Z}^d \quad (2.29)$$

In thermodynamic limit we have

$$\sigma(H_{\text{bg}} - E_{\text{bg}}, \mathbf{P}) \subset [0, \infty[\times \mathbb{R}^d, \quad (2.30)$$

and it is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{1+d}$ except for the ground state at $(0, \mathbf{0})$ and the *elementary quasiparticle spectrum* at

$$\sigma_{\text{sing}}(H_{\text{bg}} - E_{\text{bg}}, \mathbf{P}) = \{ (\omega_{\text{bg}}(\mathbf{k}), \mathbf{k}) \mid \mathbf{k} \in \mathbb{R}^d \}. \quad (2.31)$$

(2.30) is the subadditive hull of (2.31). For more details concerning the construction of the excitation spectrum we refer to [19, 26, 24].

The same Bogoliubov transformation (2.17) can be applied also to H_3^L and H_4^L , thus obtaining an interaction expressed in terms of $b_{\mathbf{k}}^*$, $b_{\mathbf{k}}$.

2.5. Positive temperatures. Let us recall basic facts about quantum physics at positive temperatures. In the back of our heads we have the algebraic description of infinitely extended systems involving C^* - and W^* -algebras [11, 12, 23], and we will sometimes use terminology developed in these frameworks. Unfortunately, the C^* -algebraic framework is problematic for interacting bosons. Maybe, one could use the C^* -algebras introduced by Buchholz [15, 16], but we have not tried them.

Suppose a quantum system is described by a Hamiltonian H and a Hilbert space $\mathcal{H}$. Thus the evolution of $A \in B(\mathcal{H})$ is given by

$$A(t) := e^{itH} A e^{-itH}. \quad (2.32)$$

It is convenient to pass to the so-called *standard representation* of $B(\mathcal{H})$. Its Hilbert space can be identified with Hilbert-Schmidt operators, denoted $B^2(\mathcal{H})$. The scalar product on $B^2(\mathcal{H})$ is given by the trace:

$$(\Phi|\Psi) := \text{Tr } \Phi^* \Psi, \quad \Phi, \Psi \in B^2(\mathcal{H}). \quad (2.33)$$

The algebra $B(\mathcal{H})$ has two commuting representations, the (linear) left one and the (antilinear) right one:

$$\pi_l(A)\Phi := A\Phi, \quad (2.34)$$

$$\pi_r(A)\Phi := \Phi A^*, \quad \Phi \in B^2(\mathcal{H}), \quad A \in B(\mathcal{H}). \quad (2.35)$$

Every normal state ϕ on $B(\mathcal{H})$ can be represented by a *vector representative*, that is a vector $\Phi \in B^2(\mathcal{H})$ such that

$$\phi(A) = (\Phi|\pi_l(A)\Phi). \quad (2.36)$$

It has a unique *standard vector representative* which is given by a positive Hilbert-Schmidt matrix. Thus for the state ϕ with the density matrix ρ_ϕ the standard vector representative is $\Phi := \sqrt{\rho_\phi}$.

At inverse temperature β the KMS state is given by the Gibbs density matrix

$$\omega_\beta(A) = \text{Tr } A \frac{e^{-\beta H}}{\text{Tr } e^{-\beta H}}, \quad A \in B(\mathcal{H}). \quad (2.37)$$

For positive temperatures, the thermal state is faithful, therefore its GNS representation coincides with the standard representation. For zero temperature the GNS representation is just the usual irreducible representation on $\mathcal{H}$. However, in order to unify the descriptions, one can use the standard representation for both positive and zero temperatures.

The standard representative of the state ω_β is

$$\Omega_\beta := \frac{e^{-\frac{\beta}{2}H}}{(\text{Tr } e^{-\beta H})^{\frac{1}{2}}}. \quad (2.38)$$

Thus

$$\omega_\beta(A) = (\Omega_\beta|\pi_l(A)\Omega_\beta). \quad (2.39)$$

The dynamics is generated by the Liouvillean

$$L := [H, \cdot], \quad (2.40)$$

so that

$$\pi_l(A(t)) = e^{itL} \pi_l(A) e^{-itL}, \quad L\Omega_\beta = 0. \quad (2.41)$$

In general, a self-adjoint operator K on $\mathcal{H}$, understood as the generator of a 1-parameter unitary group, in the standard representation should be replaced by its *standard* version

$$K^{\text{st}} := [K, \cdot]. \quad (2.42)$$

2.6. Araki-Woods representation. Let us go back to the Bose gas. Its standard representation acts on Hilbert-Schmidt operators on the Fock space, denoted

$$B^2(\Gamma_s(l^2(\Xi_L^\geq))), \quad (2.43)$$

the Hamiltonian should be replaced by the Liouvillean, and the momentum by its standard version:

$$L^L := [H^L, \cdot], \quad (2.44)$$

$$\mathbf{P}^{\text{st}, L} := [\mathbf{P}^L, \cdot]. \quad (2.45)$$

We have the Gibbs state of the full Hamiltonian ω_β . We will also use the Gibbs state for the Bogoliubov Hamiltonian $\omega_{\text{bg}, \beta}$ with the standard vector representative $\Omega_{\text{bg}, \beta}$.

We can write

$$L^L := L_{\text{bg}}^L + \sqrt{\kappa} L_3^L + \kappa L_4^L, \quad (2.46)$$

$$L_{\text{bg}}^L := [H_{\text{bg}}^L, \cdot], \quad (2.47)$$

$$L_3^L := [H_3^L, \cdot], \quad (2.48)$$

$$L_4^L := [H_4^L, \cdot]. \quad (2.49)$$

The representation on (2.43) is not very convenient for calculations. We prefer to pass to the Araki-Woods representation [2, 22] well adapted to the Bogoliubov Liouvillean L_{bg}^L .

This passage can be done in two steps. First, it is easy to see that (2.43) is naturally isomorphic to the Fock space on the doubled 1-particle space

$$\Gamma_s(l^2(\Sigma_L^\geq) \oplus l^2(\Sigma_L^\geq)). \quad (2.50)$$

Then

$$\pi_l(b(\mathbf{k})) = b_l(\mathbf{k}), \quad \pi_l(b^*(\mathbf{k})) = b_l^*(\mathbf{k}), \quad (2.51)$$

$$\pi_r(b(\mathbf{k})) = b_r(\mathbf{k}), \quad \pi_r(b^*(\mathbf{k})) = b_r^*(\mathbf{k}), \quad (2.52)$$

where we use $b_l(\mathbf{k})$, $b_l^*(\mathbf{k})$, resp. $b_r(\mathbf{k})$, $b_r^*(\mathbf{k})$ for the annihilation, creation operators acting on the left, resp. right space $l^2(\Xi_L^\geq)$. This is still not very convenient, because $\Omega_{\text{bg}, \beta}$ corresponds to a relatively complicated squeezed vector.

Then one applies an appropriate Bogoliubov transformation, obtaining the so-called Araki-Woods representation. This is still a representation on

$$\Gamma_s(l^2(\Xi_L^\geq) \oplus l^2(\Xi_L^\geq)). \quad (2.53)$$

However, in this representation the vector $\Omega_{\text{bg}, \beta}$ is mapped onto the Fock vacuum Ω , and the creation/annihilation operators are represented by $\pi_{\beta, l}$ and $\pi_{\beta, r}$ as follows:

$$b_{\beta, l}^*(\mathbf{k}) := \pi_{\beta, l}(b^*(\mathbf{k})) = (1 - e^{-\beta\omega_{\text{bg}}(\mathbf{k})})^{-\frac{1}{2}} b_l^*(\mathbf{k}) + (e^{\beta\omega_{\text{bg}}(\mathbf{k})} - 1)^{-\frac{1}{2}} b_r(\mathbf{k}), \quad (2.54)$$

$$b_{\beta, l}(\mathbf{k}) := \pi_{\beta, l}(b(\mathbf{k})) = (1 - e^{-\beta\omega_{\text{bg}}(\mathbf{k})})^{-\frac{1}{2}} b_l(\mathbf{k}) + (e^{\beta\omega_{\text{bg}}(\mathbf{k})} - 1)^{-\frac{1}{2}} b_r^*(\mathbf{k}), \quad (2.55)$$

$$b_{\beta, r}^*(\mathbf{k}) := \pi_{\beta, r}(b^*(\mathbf{k})) = (e^{\beta\omega_{\text{bg}}(\mathbf{k})} - 1)^{-\frac{1}{2}} b_l(\mathbf{k}) + (1 - e^{-\beta\omega_{\text{bg}}(\mathbf{k})})^{-\frac{1}{2}} b_r^*(\mathbf{k}), \quad (2.56)$$

$$b_{\beta, r}(\mathbf{k}) := \pi_{\beta, r}(b(\mathbf{k})) = (e^{\beta\omega_{\text{bg}}(\mathbf{k})} - 1)^{-\frac{1}{2}} b_l^*(\mathbf{k}) + (1 - e^{-\beta\omega_{\text{bg}}(\mathbf{k})})^{-\frac{1}{2}} b_r(\mathbf{k}). \quad (2.57)$$

For the following, we define

$$\rho(\mathbf{k}) := (e^{\beta\omega_{\text{bg}}(\mathbf{k})} - 1)^{-1}, \quad 1 + \rho(\mathbf{k}) = (1 - e^{-\beta\omega_{\text{bg}}(\mathbf{k})})^{-1}. \quad (2.58)$$

The free Liouvillean and the standard momentum (written in two equivalent ways) become

$$L_{\text{bg}}^L = \sum_{\mathbf{k} \in \Xi_L^{\geq}} \omega_{\text{bg}}(\mathbf{k}) \left(b_l^*(\mathbf{k}) b_l(\mathbf{k}) - b_r^*(\mathbf{k}) b_r(\mathbf{k}) \right) \quad (2.59)$$

$$= \sum_{\mathbf{k} \in \Xi_L^{\geq}} \omega_{\text{bg}}(\mathbf{k}) \left(b_{\beta,1}^*(\mathbf{k}) b_{\beta,1}(\mathbf{k}) - b_{\beta,r}^*(\mathbf{k}) b_{\beta,r}(\mathbf{k}) \right), \quad (2.60)$$

$$\mathbf{P}^{\text{st},L} = \sum_{\mathbf{k} \in \Xi_L^{\geq}} \mathbf{k} \left(b_l^*(\mathbf{k}) b_l(\mathbf{k}) - b_r^*(\mathbf{k}) b_r(\mathbf{k}) \right) \quad (2.61)$$

$$= \sum_{\mathbf{k} \in \Xi_L^{\geq}} \mathbf{k} \left(b_{\beta,1}^*(\mathbf{k}) b_{\beta,1}(\mathbf{k}) - b_{\beta,r}^*(\mathbf{k}) b_{\beta,r}(\mathbf{k}) \right). \quad (2.62)$$

In thermodynamic limit the joint spectrum of the Bogoliubov Liouvillean and the standard momentum is the whole space:

$$\sigma(L_{\text{bg}}, \mathbf{P}^{\text{st}}) = \mathbb{R}^{1+d}. \quad (2.63)$$

It has a bound state at $(0, \mathbf{0})$ corresponding to the KMS state. The singular spectrum has now two branches: the positive for “left quasiparticles” and negative for “right quasiparticles”:

$$\sigma_{\text{sing}}(L_{\text{bg}}, \mathbf{P}^{\text{st}}) = \{ (\omega_{\text{bg}}(\mathbf{k}), \mathbf{k}) \mid \mathbf{k} \in \mathbb{R}^d \} \cup \{ (-\omega_{\text{bg}}(\mathbf{k}), \mathbf{k}) \mid \mathbf{k} \in \mathbb{R}^d \}. \quad (2.64)$$

Recall that the full Hamiltonian was given in (2.6)–(2.9). It is useful to express it in terms of $b(\mathbf{k})$, $b^*(\mathbf{k})$.

The full Liouvillean is

$$L^L := L_{\text{bg}}^L + \sqrt{\kappa} L_3^L + \kappa L_4^L, \quad (2.65)$$

$$L_3^L := H_{3,1}^L - H_{3,r}^L, \quad (2.66)$$

$$L_4^L := H_{4,1}^L - H_{4,r}^L, \quad (2.67)$$

where $H_{3,1}^L, H_{4,1}^L$, resp. $H_{3,r}^L, H_{4,r}^L$ are obtained by insertion of $b_{\beta,1}(\mathbf{k})$, $b_{\beta,1}^*(\mathbf{k})$, resp. $b_{\beta,r}(\mathbf{k})$, $b_{\beta,r}^*(\mathbf{k})$ instead of $b(\mathbf{k})$, $b^*(\mathbf{k})$ in H_3^L, H_4^L .

The Liouvillean L_3^L can be written as the sum of two contributions $L_{3,1}^L + L_{3,0}^L$. The first is the one relevant for physical processes involving three quasi-particles where two of them are destroyed and one is created or vice versa, two are created and the remaining one is destroyed. It can be written as

$$L_{3,1}^L = \sum_{\substack{\mathbf{p}, \mathbf{q}, \\ \mathbf{p} + \mathbf{q} \in \Xi_L^{\geq}}} V_{\mathbf{p}, \mathbf{q}} \left(b_{\beta,1}^*(\mathbf{q} + \mathbf{p}) b_{\beta,1}(\mathbf{p}) b_{\beta,1}(\mathbf{q}) - b_{\beta,r}^*(\mathbf{q} + \mathbf{p}) b_{\beta,r}(\mathbf{p}) b_{\beta,r}(\mathbf{q}) + \text{h.c.} \right), \quad (2.68)$$

where the effective interaction is

$$V_{\mathbf{p}, \mathbf{q}} = \frac{\sqrt{\nu} \hat{v}(\mathbf{p})}{L^{\frac{d}{2}} \sqrt{\hat{v}(\mathbf{0})}} (c_{\mathbf{p}} - s_{\mathbf{p}}) (c_{\mathbf{q}} c_{\mathbf{q} + \mathbf{p}} + s_{\mathbf{q}} s_{\mathbf{q} + \mathbf{p}}) \\ + \frac{\sqrt{\nu} \hat{v}(\mathbf{p} + \mathbf{q})}{L^{\frac{d}{2}} \sqrt{\hat{v}(\mathbf{0})}} (s_{\mathbf{p} + \mathbf{q}} - c_{\mathbf{p} + \mathbf{q}}) s_{\mathbf{p}} c_{\mathbf{q}}. \quad (2.69)$$

This effective potential is related to the function $j(\mathbf{p} + \mathbf{q}; \mathbf{p}, \mathbf{q})$ in Equation (1.5), as

$$\begin{aligned} V_{\mathbf{p}, \mathbf{q}} + V_{\mathbf{q}, \mathbf{p}} &= \frac{1}{L^{\frac{d}{2}}} \sqrt{\frac{\nu}{\hat{v}(\mathbf{0})}} (\hat{v}(\mathbf{p})(c_{\mathbf{p}} - s_{\mathbf{p}})(c_{\mathbf{q}} c_{\mathbf{q}+\mathbf{p}} + s_{\mathbf{q}} s_{\mathbf{q}+\mathbf{p}}) \\ &\quad + \hat{v}(\mathbf{q})(c_{\mathbf{q}} - s_{\mathbf{q}})(c_{\mathbf{p}} c_{\mathbf{q}+\mathbf{p}} + s_{\mathbf{p}} s_{\mathbf{q}+\mathbf{p}}) \\ &\quad + \hat{v}(\mathbf{p} + \mathbf{q})(s_{\mathbf{p}+\mathbf{q}} - c_{\mathbf{q}+\mathbf{p}})(c_{\mathbf{p}} s_{\mathbf{q}} + c_{\mathbf{q}} s_{\mathbf{p}})) \\ &= \frac{1}{L^{\frac{d}{2}}} j(\mathbf{p} + \mathbf{q}; \mathbf{p}, \mathbf{q}). \end{aligned} \quad (2.70)$$

The second contribution $L_{3,0}^L$ corresponds to processes where three particles are either created or destroyed

$$\begin{aligned} L_{3,0}^L &= \sum_{\substack{\mathbf{p}, \mathbf{q}, \\ \mathbf{p}+\mathbf{q} \in \Xi_L^{\geq}}} U_{\mathbf{p}, \mathbf{q}} (b_{\beta,1}^*(-\mathbf{q} - \mathbf{p}) b_{\beta,1}^*(\mathbf{p}) b_{\beta,1}^*(\mathbf{q}) \\ &\quad - b_{\beta,r}^*(-\mathbf{q} - \mathbf{p}) b_{\beta,r}^*(\mathbf{p}) b_{\beta,r}^*(\mathbf{q}) + \text{h.c.}), \end{aligned} \quad (2.71)$$

with

$$U_{\mathbf{p}, \mathbf{q}} = \frac{\sqrt{\nu} \hat{v}(\mathbf{p})}{L^{\frac{d}{2}} \sqrt{\hat{v}(\mathbf{0})}} (c_{\mathbf{p}+\mathbf{q}} s_{\mathbf{p}} s_{\mathbf{q}} - s_{\mathbf{p}+\mathbf{q}} c_{\mathbf{p}} c_{\mathbf{q}}). \quad (2.72)$$

This effective potential is related to the function $\kappa(\mathbf{p} + \mathbf{q}, \mathbf{p}, \mathbf{q})$ in Equation (1.6) as

$$\begin{aligned} (U_{\mathbf{k}, -\mathbf{p}} + U_{-\mathbf{p}, \mathbf{k}} + U_{\mathbf{p}, \mathbf{k}-\mathbf{p}} + U_{\mathbf{p}-\mathbf{k}, \mathbf{k}} + U_{\mathbf{k}, \mathbf{p}-\mathbf{k}} + U_{\mathbf{k}-\mathbf{p}, \mathbf{p}}) = \\ = \frac{1}{L^{\frac{d}{2}}} \sqrt{\frac{\nu}{\hat{v}(\mathbf{0})}} (\hat{v}(\mathbf{k})(s_{\mathbf{k}} - c_{\mathbf{k}})(c_{\mathbf{p}} s_{\mathbf{k}-\mathbf{p}} + c_{\mathbf{k}-\mathbf{p}} s_{\mathbf{p}}) \\ + \hat{v}(\mathbf{p})(s_{\mathbf{p}} - c_{\mathbf{p}})(c_{\mathbf{k}} s_{\mathbf{k}-\mathbf{p}} + c_{\mathbf{k}-\mathbf{p}} s_{\mathbf{k}}) \\ + \hat{v}(\mathbf{k} - \mathbf{p})(s_{\mathbf{k}-\mathbf{p}} - c_{\mathbf{k}-\mathbf{p}})(c_{\mathbf{k}} s_{\mathbf{p}} + c_{\mathbf{p}} s_{\mathbf{k}})) = \frac{1}{L^{\frac{d}{2}}} \kappa(\mathbf{k}, \mathbf{p}, \mathbf{k} - \mathbf{p}). \end{aligned} \quad (2.73)$$

The Liouvillean L_4^L can be written as the sum of five terms. Two of them are of the second order:

$$\begin{aligned} L_{2,1}^L &= \frac{1}{L^d} \sum_{\mathbf{p}, \mathbf{q} \in \Xi_L^{\geq}} (\hat{v}(\mathbf{0})(c_{\mathbf{p}}^2 + s_{\mathbf{p}}^2) s_{\mathbf{q}}^2 b_{\beta,1}^*(\mathbf{p}) b_{\beta,1}(\mathbf{p}) \\ &\quad + \hat{v}(\mathbf{q} - \mathbf{p})((c_{\mathbf{p}}^2 + s_{\mathbf{p}}^2) s_{\mathbf{q}}^2 + 2c_{\mathbf{p}} c_{\mathbf{q}} s_{\mathbf{p}} s_{\mathbf{q}}) b_{\beta,1}^*(\mathbf{p}) b_{\beta,1}(\mathbf{p}) - (l \leftrightarrow r)), \end{aligned} \quad (2.74)$$

$$\begin{aligned} L_{2,0}^L &= -\frac{1}{2L^d} \sum_{\mathbf{p}, \mathbf{q} \in \Xi_L^{\geq}} (\hat{v}(\mathbf{0}) 2c_{\mathbf{p}} s_{\mathbf{p}} s_{\mathbf{q}}^2 \\ &\quad + \hat{v}(\mathbf{q} - \mathbf{p})((s_{\mathbf{p}}^2 + c_{\mathbf{p}}^2) c_{\mathbf{q}} s_{\mathbf{q}} + 2c_{\mathbf{p}} s_{\mathbf{p}} s_{\mathbf{q}}^2)) (b_{\beta,1}^*(\mathbf{p}) b_{\beta,1}^*(-\mathbf{p}) \\ &\quad + \text{h.c.} - (l \leftrightarrow r)). \end{aligned} \quad (2.75)$$

The remaining three are of the fourth order:

$$L_{4,2}^L = \frac{1}{2L^d} \sum_{\substack{\mathbf{p}, \mathbf{q}, \\ \mathbf{p}+\mathbf{k}, \mathbf{q}-\mathbf{k} \in \Xi_L^\geq}} [\hat{v}(\mathbf{k})(c_{\mathbf{k}+\mathbf{p}}c_{\mathbf{q}-\mathbf{k}}c_{\mathbf{q}}c_{\mathbf{p}} + s_{\mathbf{p}+\mathbf{k}}s_{\mathbf{q}-\mathbf{k}}s_{\mathbf{p}}s_{\mathbf{q}} \\ + 2c_{\mathbf{k}+\mathbf{p}}c_{\mathbf{p}}s_{\mathbf{q}}s_{\mathbf{q}-\mathbf{k}}) + 2\hat{v}(\mathbf{p}+\mathbf{q})c_{\mathbf{q}}c_{\mathbf{q}-\mathbf{k}}s_{\mathbf{p}}s_{\mathbf{p}+\mathbf{k}}] \\ \times (b_{\beta,1}^*(\mathbf{k}+\mathbf{p})b_{\beta,1}^*(\mathbf{q}-\mathbf{k})b_{\beta,1}(\mathbf{q})b_{\beta,1}(\mathbf{p}) - (l \leftrightarrow r)), \quad (2.76)$$

$$L_{4,1}^L = -\frac{1}{L^d} \sum_{\substack{\mathbf{p}, \mathbf{q}, \\ \mathbf{p}+\mathbf{k}, \mathbf{q}-\mathbf{k} \in \Xi_L^\geq}} \hat{v}(\mathbf{k}) \\ \times (c_{\mathbf{k}+\mathbf{p}}c_{\mathbf{q}-\mathbf{k}}c_{\mathbf{p}}s_{\mathbf{q}}b_{\beta,1}^*(\mathbf{p}+\mathbf{k})b_{\beta,1}^*(\mathbf{q}-\mathbf{k})b_{\beta,1}^*(-\mathbf{q})b_{\beta,1}(\mathbf{p}) \\ + c_{\mathbf{p}+\mathbf{k}}s_{\mathbf{q}-\mathbf{k}}s_{\mathbf{q}}s_{\mathbf{p}}b_{\beta,1}^*(\mathbf{p}+\mathbf{k})b_{\beta,1}^*(-\mathbf{p})b_{\beta,1}^*(\mathbf{q})b_{\beta,1}(\mathbf{k}-\mathbf{q}) \\ + \text{h.c.} - (l \leftrightarrow r)) \quad (2.77)$$

$$L_{4,0}^L = \frac{1}{2L^d} \sum_{\substack{\mathbf{p}, \mathbf{q}, \\ \mathbf{p}+\mathbf{k}, \mathbf{q}-\mathbf{k} \in \Xi_L^\geq}} \hat{v}(\mathbf{k})c_{\mathbf{k}+\mathbf{p}}c_{\mathbf{q}-\mathbf{k}}s_{\mathbf{q}}s_{\mathbf{p}} \\ \times (b_{\beta,1}^*(\mathbf{p}+\mathbf{k})b_{\beta,1}^*(\mathbf{q}-\mathbf{k})b_{\beta,1}^*(-\mathbf{p})b_{\beta,1}^*(-\mathbf{q}) + \text{h.c.} - (l \leftrightarrow r)). \quad (2.78)$$

In the above we labeled with $(l \leftrightarrow r)$ the terms which can be obtained from the ones explicitly written by exchanging left and right particles, which is equivalent to exchanging the labels l and r .

The KMS vector for the Bogoliubov Liouvillean, in the Araki-Woods representation, is simply the vacuum Ω . By general theory, we are guaranteed that the perturbed KMS state exists and is an eigenvector of the full Liouvillean with eigenvalue 0. It can be obtained by perturbation theory. All terms in the perturbation expansion for its energy will give 0. In particular, at the order κ for a fixed L we have

$$0 = (\Omega | L_4^L \Omega) - \lim_{\epsilon \searrow 0} \left(\Omega | L_3^L (L_{\text{bg}}^L - i\epsilon)^{-1} L_3^L \Omega \right). \quad (2.79)$$

2.7. Elementary quasiparticle from joint energy-momentum spectrum. If one could make sense of the Hamiltonian or Liouvillean with positive ν in thermodynamic limit, then one should expect that the joint energy-momentum spectrum is absolutely continuous everywhere, without singular shells, except for the ground state at zero temperature and the KMS vector at positive temperature. In particular, the thermodynamic limit probably destroys the singular shells (2.31) and (2.64).

Thus even if the Hamiltonian H and the Liouvillean L can be defined in thermodynamic limit, rigorously it is not clear what are quasiparticles beyond the Bogoliubov approximation. Still, physicists seem to believe that quasiparticles are a useful concept.

At zero temperature the interacting Hamiltonian is bounded from below and one can expect that it makes sense to speak about the infimum of the energy-momentum spectrum in thermodynamic limit, as described in [19]. Typically, in the unperturbed Bogoliubov Hamiltonian this infimum does not coincide with the quasiparticle spectrum (2.31), even for small momenta. So in the interacting case this infimum is not a good definition of the quasiparticle spectrum. Anyway, for zero temperature at least a certain interesting purely spectral information seems to survive thermodynamic limit.

For positive temperatures, the joint spectrum of the interacting Liouvillean and the standard momentum in thermodynamic limit is almost certainly the full $\mathbb{R}^{1+d}$, so from the spectrum we cannot extract anything interesting.

2.8. Decay of one-quasiparticle states. As we mentioned above, for the pure Bogoliubov Liouvillean in the Araki-Woods representation the standard vector representative of the KMS state is the Fock vacuum Ω . Switching on the interaction modifies the KMS vector, but it remains an eigenvector with eigenvalue $(0, \mathbf{0})$.

For the pure Bogoliubov Liouvillean the one-quasiparticle state of momentum $\mathbf{k}$ has the standard vector representative, which in the Araki-Woods representation is given by

$$b_l^*(\mathbf{k})b_r^*(\mathbf{k})\Omega. \quad (2.80)$$

It is an eigenstate of $(L_{\text{bg}}^L, \mathbf{P}^{\text{st},L})$ with eigenvalue $(0, \mathbf{0})$.

After switching the interaction in finite volume we expect that perturbation theory will yield perturbed eigenstates of the Liouvillean. However, these eigenstates will not survive thermodynamic limit. In fact, we will find that already at the lowest nontrivial order of perturbation theory we will obtain a nonzero imaginary contribution.

The computation is complicated by the fact that we are dealing with two limits: small coupling constant κ and large size L . The value of the shift of the eigenvalue at the order κ^1 will be computed from the formula

$$\xi(\mathbf{k}) := \lim_{\epsilon \searrow 0} \lim_{L \rightarrow \infty} \left\{ \left(b_l^*(\mathbf{k})b_r^*(\mathbf{k})\Omega | L_4^L b_l^*(\mathbf{k})b_r^*(\mathbf{k})\Omega \right) \right. \quad (2.81)$$

$$\left. - \left(L_3^L b_l^*(\mathbf{k})b_r^*(\mathbf{k})\Omega | (L_{\text{bg}}^L - i\epsilon)^{-1} L_3^L b_l^*(\mathbf{k})b_r^*(\mathbf{k})\Omega \right) \right. \quad (2.82)$$

$$\left. - (\Omega | L_4^L \Omega) + (L_3^L \Omega | (L_{\text{bg}}^L - i\epsilon)^{-1} L_3^L \Omega) \right\}. \quad (2.83)$$

We will see that the shift $\xi(\mathbf{k})$ is purely imaginary.

Above, (2.81)+(2.82) is the naive expression for the shift of the eigenvalue in thermodynamic limit. In fact, for finite L the shift is given by

$$\lim_{\epsilon \searrow 0} \left\{ \left(b_l^*(\mathbf{k})b_r^*(\mathbf{k})\Omega | L_4^L b_l^*(\mathbf{k})b_r^*(\mathbf{k})\Omega \right) \right. \quad (2.84)$$

$$\left. - \left(L_3^L b_l^*(\mathbf{k})b_r^*(\mathbf{k})\Omega | (L_{\text{bg}}^L - i\epsilon)^{-1} L_3^L b_l^*(\mathbf{k})b_r^*(\mathbf{k})\Omega \right) \right\} \quad (2.85)$$

However, we need to subtract from this expression the term (2.83), which for fixed L and $\epsilon \searrow 0$ goes to zero by (2.79), is however extensive in volume.

Let us check that the subtraction of (2.83) is appropriate. Introduce the shorthand $b_{lr}(\mathbf{k}) := b_l(\mathbf{k})b_r(\mathbf{k})$. The expression from (2.84) and (2.85) can be transformed as follows:

$$(b_{lr}^*(\mathbf{k})\Omega | L_4^L b_{lr}^*(\mathbf{k})\Omega) - (b_{lr}^*(\mathbf{k})\Omega | L_3^L (L_{\text{bg}}^L - i\epsilon)^{-1} L_3^L b_{lr}^*(\mathbf{k})\Omega) \quad (2.86)$$

$$= (\Omega | L_4^L \Omega) - (\Omega | L_3^L (L_{\text{bg}}^L - i\epsilon)^{-1} L_3^L \Omega) \quad (2.87)$$

$$\begin{aligned} &+ (\Omega | [b_{lr}(\mathbf{k}), [L_4^L, b_{lr}^*(\mathbf{k})]] \Omega) \\ &- (\Omega | [b_{lr}(\mathbf{k}), L_3^L] (L_{\text{bg}}^L - i\epsilon)^{-1} [L_3^L, b_{lr}^*(\mathbf{k})] \Omega) \\ &- (\Omega | [b_{lr}^*(\mathbf{k}), L_3^L] (L_{\text{bg}}^L - i\epsilon)^{-1} [L_3^L, b_{lr}(\mathbf{k})] \Omega) \\ &- (\Omega | L_3^L (L_{\text{bg}}^L - i\epsilon)^{-1} [b_{lr}(\mathbf{k}), [L_3^L, b_{lr}^*(\mathbf{k})]] \Omega) \\ &- (\Omega | [b_{lr}^*(\mathbf{k}), [L_3^L, b_{lr}(\mathbf{k})]] (L_{\text{bg}}^L - i\epsilon)^{-1} L_3^L \Omega). \end{aligned}$$

To derive this we used

$$[b_{\text{lr}}(\mathbf{k}), b_{\text{lr}}^*(\mathbf{k})] = b_{\text{r}}^*(\mathbf{k})b_{\text{r}}(\mathbf{k}) + b_{\text{l}}^*(\mathbf{k})b_{\text{l}}(\mathbf{k}) + 1, \quad (2.88)$$

$$b_{\text{lr}}^*(\mathbf{k})(L_{\text{bg}}^L - i\epsilon)^{-1} = (L_{\text{bg}}^L - i\epsilon)^{-1}b_{\text{lr}}^*(\mathbf{k}), \quad (2.89)$$

$$b_{\text{lr}}(\mathbf{k})(L_{\text{bg}}^L - i\epsilon)^{-1} = (L_{\text{bg}}^L - i\epsilon)^{-1}b_{\text{lr}}(\mathbf{k}). \quad (2.90)$$

In the formula for (2.86), by (2.79), the term (2.87) goes to zero as $\epsilon \rightarrow 0$. At the same time, for a fixed $\epsilon > 0$, it is large, diverging as $L \rightarrow \infty$. Therefore we need to subtract it. All other terms contain a commutator, so that one can hope that they do not blow up as $L \rightarrow \infty$.

2.9. Two-point functions. Let us go back to the abstract setting used in Subsection 2.5. Let $A_1, \dots, A_n \in B(\mathcal{H})$ be operators. Physicists often consider *n-point correlation functions* defined as

$$i^{\frac{n}{2}}\omega_{\beta}\left(\text{T}\left(A_n(t_n)\cdots A_1(t_1)\right)\right), \quad (2.91)$$

where $\text{T}(\cdots)$ denotes the time ordering [36]. (The coefficient $i^{\frac{n}{2}}$ follows one of the possible conventions, which is convenient in applications to the Bose gas). Of special importance are two-point functions

$$i\omega_{\beta}\left(\text{T}\left(A_2(t_2)A_1(t_1)\right)\right) \quad (2.92)$$

$$= i\theta(t_2 - t_1)\omega_{\beta}(A_2(t_2)A_1(t_1)) + i\theta(t_1 - t_2)\omega_{\beta}(A_1(t_1)A_2(t_2)) \quad (2.93)$$

$$= i\theta(t_2 - t_1)(\Omega_{\beta}|A_2 e^{i(t_1-t_2)L} A_1\Omega_{\beta}) + i\theta(t_1 - t_2)(\Omega_{\beta}|A_1 e^{i(t_2-t_1)L} A_2\Omega_{\beta}). \quad (2.94)$$

where in (2.94) we use the standard representation and we omitted π_1 . Note that (2.94) depends only on $t := t_2 - t_1$, so that we can define

$$G_{\beta}(A_2, A_1, t) := (2.94). \quad (2.95)$$

Let $E \in \mathbb{R}$. The two-point functions in the energy representation are

$$\begin{aligned} G_{\beta}(A_2, A_1; E) &:= \int G_{\beta}(A_2, A_1, t) e^{-itE} dt \\ &= \lim_{\epsilon \searrow 0} i \int_0^{\infty} (\Omega_{\beta}|A_2 e^{-itL - \epsilon t - itE} A_1\Omega_{\beta}) dt \\ &\quad + \lim_{\epsilon \searrow 0} i \int_0^{\infty} (\Omega_{\beta}|A_1 e^{-itL - \epsilon t + itE} A_2\Omega_{\beta}) dt \\ &= (\Omega_{\beta}|A_2(L - i0 + E)^{-1} A_1\Omega_{\beta}) + (\Omega_{\beta}|A_1(L - i0 - E)^{-1} A_2\Omega_{\beta}). \end{aligned} \quad (2.97)$$

(2.97) is sometimes called the *Lehmann representation*.

As a side remark let us mention that it is often possible to simplify the above formula if we have time reversal invariance. More precisely, assume that there exists an antilinear involution $\Phi \mapsto \bar{\Phi}$, such that $H = \bar{H}$, and hence $\bar{L} = L$ and $\bar{\Omega}_{\beta} = \Omega_{\beta}$. Let A_1, A_2 be also real, that is, $\bar{A}_1 = A_1$, $\bar{A}_2 = A_2$. Then the second term in (2.97) can be written as

$$(\Omega_{\beta}|A_2^*(L - i0 - E)^{-1} A_1^*\Omega_{\beta}). \quad (2.98)$$

Thus, if A_1 and A_2 are self-adjoint, then

$$G_{\beta}(A_2, A_1; E) = (\Omega_{\beta}|A_2 2L((L - i0)^2 - E^2)^{-1} A_1\Omega_{\beta}). \quad (2.99)$$

2.10. Elementary quasiparticle spectrum from two-point functions. Physicists believe that quasiparticles can be seen in correlation functions. The simplest choice for observables A_1, A_2 in these correlation functions seems to be $a_{\mathbf{k}}^*, a_{\mathbf{k}}$. This choice is perhaps not the most convenient experimentally, however it seems mathematically the most natural and was adopted in classic physics papers, e.g. by Beliaev and Hugenholtz-Pines. Thus we define

$$\begin{bmatrix} G_{11}(E, \mathbf{k}) & G_{12}(E, \mathbf{k}) \\ G_{21}(E, \mathbf{k}) & G_{22}(E, \mathbf{k}) \end{bmatrix} \quad (2.100)$$

$$:= i \int dt e^{-itE} \begin{bmatrix} \omega_{\beta}(T(a_{\mathbf{k}}^*(t)a_{\mathbf{k}}(0))) & \omega_{\beta}(T(a_{\mathbf{k}}^*(t)a_{-\mathbf{k}}^*(0))) \\ \omega_{\beta}(T(a_{-\mathbf{k}}(t)a_{\mathbf{k}}(0))) & \omega_{\beta}(T(a_{-\mathbf{k}}(t)a_{-\mathbf{k}}^*(0))) \end{bmatrix}. \quad (2.101)$$

A linear transformation connects $a_{\mathbf{k}}^*, a_{\mathbf{k}}$ with $b_{\mathbf{k}}^*, b_{\mathbf{k}}$. Therefore, one can instead consider

$$i \int dt e^{-itE} \begin{bmatrix} \omega_{\beta}(T(b_{\mathbf{k}}^*(t)b_{\mathbf{k}}(0))) & \omega_{\beta}(T(b_{\mathbf{k}}^*(t)b_{-\mathbf{k}}^*(0))) \\ \omega_{\beta}(T(b_{-\mathbf{k}}(t)b_{\mathbf{k}}(0))) & \omega_{\beta}(T(b_{-\mathbf{k}}(t)b_{-\mathbf{k}}^*(0))) \end{bmatrix}. \quad (2.102)$$

For the pure Bogoliubov Hamiltonian, the evaluation of (2.102) can be carried out exactly via the *Lehmann representation* (2.97). As an example, we show the computation of the first diagonal entry

$$\begin{aligned} & i \int dt e^{-itE} \omega_{\beta}(T(b_{\mathbf{k}}^*(t)b_{\mathbf{k}}(0))) \\ &= (\Omega | b_{\beta,1}^*(\mathbf{k})(L_{\text{bg}} - i0 + E)^{-1} b_{\beta,1}(\mathbf{k}) \Omega) + (\Omega | b_{\beta,1}(\mathbf{k})(L_{\text{bg}} - i0 - E)^{-1} b_{\beta,1}^*(\mathbf{k}) \Omega) \\ &= -\rho(\mathbf{k}) \frac{1}{\omega_{\text{bg}}(\mathbf{k}) - i0 - E} + (1 + \rho(\mathbf{k})) \frac{1}{\omega_{\text{bg}}(\mathbf{k}) - i0 - E} \\ &= \frac{1}{\omega_{\text{bg}}(\mathbf{k}) - i0 - E}, \end{aligned} \quad (2.103)$$

$$(2.104)$$

where in the second line (2.103) we have written the state in the Araki-Woods representation. The final result is the following one

$$\begin{bmatrix} (\omega_{\text{bg}}(\mathbf{k}) - i0 - E)^{-1} & 0 \\ 0 & (\omega_{\text{bg}}(\mathbf{k}) - i0 + E)^{-1} \end{bmatrix} \quad (2.105)$$

In particular, for E approaching the real line from above, the imaginary part of (2.105) will be proportional to the delta function along the quasiparticle dispersion relation:

$$i\pi \begin{bmatrix} \delta(\omega_{\text{bg}}(\mathbf{k}) - E) & 0 \\ 0 & \delta(\omega_{\text{bg}}(\mathbf{k}) + E) \end{bmatrix} \quad (2.106)$$

When we compute the dispersion relation in the interacting case in perturbation theory we expect to obtain a complex dispersion relation with a negative imaginary part $\mathbf{k} \mapsto \omega(\mathbf{k})$. Then we can omit $-i0$ and the two-point functions look like

$$\begin{aligned} & \begin{bmatrix} G_{11}(E, \mathbf{k}) & G_{12}(E, \mathbf{k}) \\ G_{21}(E, \mathbf{k}) & G_{22}(E, \mathbf{k}) \end{bmatrix} \\ &= Z^{\pm}(E, \mathbf{k}) \frac{1}{\omega(\mathbf{k}) \mp E} + \text{regular}, \quad \text{near } \pm \omega_{\text{bg}}(\mathbf{k}) = E. \end{aligned} \quad (2.107)$$

Above, $Z^{\pm}(E, \mathbf{k})$ are some slowly varying matrices. Thus (2.106) should be replaced by a “ridge of mountains” along the quasiparticle dispersion relation, perhaps slightly shifted in the real direction. The width of this “ridge” is determined by $\text{Im}\omega(\mathbf{k})$. In the lowest order of perturbation theory, the shift of $\omega(\mathbf{k})$ includes

the Feynman-Hellman term for L_4 and the Fermi Golden Rule term for L_3 . The shift of the dispersion relation should be computed from the formula

$$\delta(\mathbf{k}) := \lim_{\epsilon \searrow 0} \lim_{L \rightarrow \infty} \left\{ \left(b_1^*(\mathbf{k}) \Omega | L_4^L b_1^*(\mathbf{k}) \Omega \right) \right. \quad (2.108)$$

$$\left. - \left(L_3^L b_1^*(\mathbf{k}) \Omega | (L_{\text{bg}}^L - \omega_{\text{bg}}(\mathbf{k}) - i\epsilon)^{-1} L_3^L b_1^*(\mathbf{k}) \Omega \right) \right. \quad (2.109)$$

$$\left. - (\Omega | L_4^L \Omega) + (L_3^L \Omega | (L_{\text{bg}}^L - i\epsilon)^{-1} L_3^L \Omega) \right\} \quad (2.110)$$

Again, from the naive expression (2.108)+(2.109) we need to subtract the extensive quantity (2.110).

The formula for $\delta(\mathbf{k})$ is suggested by the following computation:

$$(b_1^*(\mathbf{k}) \Omega | L_4^L b_1^*(\mathbf{k}) \Omega) - (b_1^*(\mathbf{k}) \Omega | L_3^L (L_{\text{bg}}^L - \omega_{\text{bg}}(\mathbf{k}) - i\epsilon)^{-1} L_3^L b_1^*(\mathbf{k}) \Omega) \quad (2.111)$$

$$= (\Omega | L_4^L \Omega) - \left(\Omega | L_3^L (L_{\text{bg}}^L - i\epsilon)^{-1} L_3^L \Omega \right) \quad (2.112)$$

$$\begin{aligned} &+ (\Omega | [b_1(\mathbf{k}), [L_4^L, b_1^*(\mathbf{k})]] \Omega) \\ &- (\Omega | [b_1(\mathbf{k}), L_3^L] (L_{\text{bg}}^L - \omega_{\text{bg}}(\mathbf{k}) - i\epsilon)^{-1} [L_3^L, b_1^*(\mathbf{k})] \Omega) \\ &- (\Omega | [b_1^*(\mathbf{k}), L_3^L] (L_{\text{bg}}^L + \omega_{\text{bg}}(\mathbf{k}) - i\epsilon)^{-1} [L_3^L, b_1(\mathbf{k})] \Omega) \\ &- (\Omega | L_3^L (L_{\text{bg}}^L - i\epsilon)^{-1} [b_1(\mathbf{k}), [L_3^L, b_1^*(\mathbf{k})]] \Omega) \\ &- (\Omega | [b_1^*(\mathbf{k}), [L_3^L, b_1(\mathbf{k})]] (L_{\text{bg}}^L - i\epsilon)^{-1} L_3^L \Omega). \end{aligned}$$

To derive this we used

$$[b_1(\mathbf{k}), b_1^*(\mathbf{k})] = 1, \quad (2.113)$$

$$b_1^*(\mathbf{k}) (L_{\text{bg}}^L + \omega_{\text{bg}}(\mathbf{k}) - i\epsilon)^{-1} = (L_{\text{bg}}^L - i\epsilon)^{-1} b_1^*(\mathbf{k}), \quad (2.114)$$

$$b_1(\mathbf{k}) (L_{\text{bg}}^L - \omega_{\text{bg}}(\mathbf{k}) - i\epsilon)^{-1} = (L_{\text{bg}}^L - i\epsilon)^{-1} b_1(\mathbf{k}). \quad (2.115)$$

Now (2.111) consists of an extensive quantity (2.112), and a few terms involving commutators, which should have a better behavior as $L \rightarrow \infty$. (2.112) goes to zero as $\epsilon \searrow 0$ by (2.79). It needs to be subtracted.

Using the above formulas, one can verify that in the thermodynamic limit

$$\lim_{|\mathbf{k}| \rightarrow 0} |\text{Re} \delta(\mathbf{k})| = +\infty.$$

This is a consequence of the naive c -number, which misses relevant terms in the construction of the effective Liouvillean. This problem is solved in [27] in the zero temperature setting. Nonetheless, $\text{Im} \delta(\mathbf{k})$ is finite and it is consistent with the imaginary shift of Subsection 2.8:

$$\text{Im} \xi(\mathbf{k}) = 2 \text{Im} \delta(\mathbf{k}). \quad (2.116)$$

2.11. Computation of matrix elements for the Liouvillean. As we are interested in finite temperature computations, we will work in the Araki-Woods representation, where the interaction is implemented by the Liouvillean L_3^L and L_4^L .

We begin by computing the relevant matrix elements for $\xi(\mathbf{k})$. The final result is summarized in the following

Proposition 2.1. Fix $\nu_0 \in]0, +\infty[$ and suppose the potential satisfies Assumptions (2.2a)–(3.5b) for all $\nu \in]0, \nu_0]$. Then, we have the following

$$\begin{aligned} \xi(\mathbf{k}) = & -i\pi \int \frac{d\mathbf{p}}{(2\pi)^d} j(\mathbf{k}; \mathbf{p}, \mathbf{p} - \mathbf{k})^2 \rho(\mathbf{p}) \rho(\mathbf{p} - \mathbf{k}) \rho(\mathbf{k}) \\ & \times \left(1 - e^{\beta \frac{\omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p} - \mathbf{k}) + \omega_{\text{bg}}(\mathbf{k})}{2}} \right)^2 \delta(\omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p} - \mathbf{k}) - \omega_{\text{bg}}(\mathbf{k})) \end{aligned} \quad (2.117)$$

$$\begin{aligned} & -i2\pi \int \frac{d\mathbf{p}}{(2\pi)^d} j(\mathbf{p} - \mathbf{k}; \mathbf{p}, \mathbf{k})^2 \rho(\mathbf{p}) \rho(\mathbf{p} - \mathbf{k}) \rho(\mathbf{k}) \\ & \times \left(e^{\beta \frac{\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p} - \mathbf{k})}{2}} - e^{\frac{\beta}{2} \omega_{\text{bg}}(\mathbf{p})} \right)^2 \delta(\omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{k}) - \omega_{\text{bg}}(\mathbf{p} - \mathbf{k})), \end{aligned} \quad (2.118)$$

where $j(\mathbf{k}; \mathbf{p}, \mathbf{q})$ is the function defined in (1.5).

Remark 2.2. Note that Integrals (2.117) and (2.118) coincide with the ones in the introduction, (1.7) and (1.8) respectively, apart from a factor of 2. This additional factor appears in (2.117), (2.118) because the vector state $b_{\text{r}}^*(\mathbf{k})b_{\text{l}}^*(\mathbf{k})\Omega$ describes processes with two incoming quasi-particles, a left one and a right one.

Proof. First of all, we note the following: L_4^L and L_3^L are antisymmetric under the exchange $l \leftrightarrow r$ while the vectors Ω and $b_{\text{lr}}^*\Omega$ are symmetric with respect to the same exchange. Thus, we deduce immediately that

$$(\Omega | L_4^L \Omega) = (b_{\text{lr}}^* \Omega | L_4^L b_{\text{lr}}^* \Omega) = 0, \quad (2.119)$$

and the same holds for L_3^L matrix elements. Thus, we only need to consider three vectors

$$[b_{\text{lr}}^*(\mathbf{k}), L_3^L] \Omega, \quad [b_{\text{lr}}(\mathbf{k}), L_3^L] \Omega, \quad [b_{\text{lr}}(\mathbf{k}), [b_{\text{lr}}^*(\mathbf{k}), L_3^L]] \Omega, \quad (2.120)$$

and the matrix elements of $(L_{\text{bg}} - i\epsilon)^{-1}$ with respect to these. With the constraints $\mathbf{p}, \mathbf{q}, \mathbf{p} + \mathbf{q} \neq \mathbf{0}$, the only non-zero vectors which might appear are all of the following form

$$[b_{\text{lr}}^*(\mathbf{k}), b_{\text{l}}^*(\mathbf{p})b_{\text{l}}^*(\mathbf{q})b_{\text{l}}(\mathbf{p} + \mathbf{q})] \Omega = -\delta_{\mathbf{k}, \mathbf{p} + \mathbf{q}} b_{\text{r}}^*(\mathbf{k})b_{\text{l}}^*(\mathbf{k} - \mathbf{p})b_{\text{l}}^*(\mathbf{p})\Omega \quad (2.121)$$

$$[b_{\text{lr}}^*(\mathbf{k}), b_{\text{l}}^*(\mathbf{p} + \mathbf{q})b_{\text{r}}^*(\mathbf{q})b_{\text{l}}(\mathbf{p})] \Omega = -\delta_{\mathbf{k}, \mathbf{p}} b_{\text{l}}^*(\mathbf{q} + \mathbf{k})b_{\text{r}}^*(\mathbf{q})b_{\text{r}}^*(\mathbf{k})\Omega \quad (2.122)$$

$$[b_{\text{lr}}^*(\mathbf{k}), b_{\text{r}}^*(\mathbf{q} + \mathbf{p})b_{\text{r}}^*(-\mathbf{q})b_{\text{l}}(-\mathbf{p})] \Omega = -\delta_{-\mathbf{p}, \mathbf{k}} b_{\text{r}}(\mathbf{k})b_{\text{r}}^*(\mathbf{q} - \mathbf{k})b_{\text{r}}^*(-\mathbf{q})\Omega \quad (2.123)$$

plus their $l \leftrightarrow r$ counterparts. All of (2.121)–(2.123) are eigenvectors of $(L_{\text{bg}} - i\epsilon)^{-1}$ and they are orthogonal to each other. In particular, we have

$$[b_{\text{lr}}(\mathbf{k}), [b_{\text{lr}}^*(\mathbf{k}), L_3^L]] \Omega = [b_{\text{lr}}(\mathbf{k}), L_3^L] \Omega = 0. \quad (2.124)$$

Thus, we see that the only matrix element which will contribute to $\xi(\mathbf{k})$ is given by

$$-([L_3^L, b_{\text{lr}}^*(\mathbf{k})] \Omega | (L_{\text{bg}}^L - i\epsilon)^{-1} [L_3^L, b_{\text{lr}}^*(\mathbf{k})] \Omega). \quad (2.125)$$

Expanding the operators $b_{\beta, \text{l}}, b_{\beta, \text{r}}$ in terms of $b_{\text{l}}, b_{\text{r}}$ and using Equations (2.121)–(2.123), we see that

$$\begin{aligned} & [L_{3,1}^L, b_{\text{lr}}^*(\mathbf{k})] \Omega \\ = & \sum_{\mathbf{p} \neq \mathbf{0}, \mathbf{p} \neq \mathbf{k}} V_{\mathbf{p}, \mathbf{k} - \mathbf{p}} \left(\sqrt{(1 + \rho(\mathbf{p}))(1 + \rho(\mathbf{p} - \mathbf{k}))(1 + \rho(\mathbf{k}))} \right. \\ & \left. - \sqrt{\rho(\mathbf{p})\rho(\mathbf{p} - \mathbf{k})\rho(\mathbf{k})} \right) b_{\text{r}}^*(\mathbf{k})b_{\text{l}}^*(\mathbf{p})b_{\text{l}}^*(\mathbf{k} - \mathbf{p})\Omega - (l \leftrightarrow r) \end{aligned} \quad (2.126)$$

$$\begin{aligned} & + \sum_{\mathbf{p}, \mathbf{p} - \mathbf{k} \in \Xi_L^{\geq}} (V_{\mathbf{k}, \mathbf{p}} + V_{\mathbf{p}, \mathbf{k}}) \left(\sqrt{(1 + \rho(\mathbf{p} - \mathbf{k}))(1 + \rho(\mathbf{k}))\rho(\mathbf{p})} \right. \\ & \left. - \sqrt{\rho(\mathbf{k})\rho(\mathbf{p} - \mathbf{k})(1 + \rho(\mathbf{p}))} \right) b_{\text{l}}^*(\mathbf{k} - \mathbf{p})b_{\text{r}}^*(\mathbf{k})b_{\text{r}}^*(-\mathbf{p})\Omega - (l \leftrightarrow r) \end{aligned} \quad (2.127)$$

and

$$\begin{aligned}
& [L_{3,0}^L, b_{\text{lr}}^*(\mathbf{k})]\Omega \\
&= \sum_{\mathbf{p}, \mathbf{p}-\mathbf{k} \in \Xi_L^{\geq}} (U_{\mathbf{k},-\mathbf{p}} + U_{-\mathbf{p},\mathbf{k}} + U_{\mathbf{p},\mathbf{k}-\mathbf{p}}) \left(\sqrt{(1+\rho(\mathbf{k}))\rho(\mathbf{p})\rho(\mathbf{p}-\mathbf{k})} \right. \\
&\quad \left. - \sqrt{\rho(\mathbf{k})(1+\rho(\mathbf{p}-\mathbf{k}))(1+\rho(\mathbf{p}))} \right) b_{\text{r}}^*(\mathbf{k})b_{\text{r}}^*(\mathbf{p}-\mathbf{k})b_{\text{r}}^*(-\mathbf{p})\Omega - (1 \leftrightarrow \text{r}). \quad (2.128)
\end{aligned}$$

Now, the vectors appearing in the sum in (2.128) are eigenvectors of $(L_{\text{bg}} - i\epsilon)^{-1}$ with eigenvalues

$$-(\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p}-\mathbf{k}) + i\epsilon)^{-1},$$

while the terms labeled by $(1 \leftrightarrow \text{r})$ have eigenvalues

$$(\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p}-\mathbf{k}) - i\epsilon)^{-1}.$$

Thus, we obtain

$$\begin{aligned}
& ([L_{3,0}^L, b_{\text{lr}}^*(\mathbf{k})]\Omega)(L_{\text{bg}}^L - i\epsilon)^{-1}[L_{3,0}^L, b_{\text{lr}}^*(\mathbf{k})]\Omega) \\
&= \frac{1}{2} \sum_{\mathbf{p}, \mathbf{p}-\mathbf{k} \in \Xi_L^{\geq}} (U_{\mathbf{k},-\mathbf{p}} + U_{-\mathbf{p},\mathbf{k}} + U_{\mathbf{p},\mathbf{k}-\mathbf{p}} + U_{\mathbf{p}-\mathbf{k},\mathbf{k}} + U_{\mathbf{k},\mathbf{p}-\mathbf{k}} + U_{\mathbf{k}-\mathbf{p},\mathbf{p}})^2 \quad (2.129) \\
&\quad \times \left(\sqrt{(1+\rho(\mathbf{k}))\rho(\mathbf{p})\rho(\mathbf{p}-\mathbf{k})} - \sqrt{\rho(\mathbf{k})(1+\rho(\mathbf{p}-\mathbf{k}))(1+\rho(\mathbf{p}))} \right)^2 \\
&\quad \times \frac{2i\epsilon}{(\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p}-\mathbf{k}))^2 + \epsilon^2}, \quad (2.130)
\end{aligned}$$

where in the first line (2.129) we have symmetrized the effective potential on account of the symmetry $\mathbf{p} \leftrightarrow \mathbf{k}-\mathbf{p}$ of the whole expression. As the effective potential $U_{\mathbf{q},\mathbf{p}}$ contains a factor $\frac{1}{L^{\frac{d}{2}}}$, the thermodynamic limit of (2.130) does converge as $L \rightarrow +\infty$ to a finite integral. After the thermodynamic limit, we can consider the $\epsilon \searrow 0$ one. In this limit, the last factor in (2.130) converges in distribution to $2\pi i \delta(\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p}-\mathbf{k})) = 0$. Moreover, from the expressions (2.126), (2.127) and (2.128) and a simple application of Wick's theorem, we see that

$$([L_{3,1}^L, b_{\text{lr}}^*(\mathbf{k})]\Omega)(L_{\text{bg}}^L - i\epsilon)^{-1}[L_{3,0}^L, b_{\text{lr}}^*(\mathbf{k})]\Omega = 0. \quad (2.131)$$

Hence, $L_{3,0}^L$ does not contribute to the damping. All that is left is to compute

$$\begin{aligned}
& ([L_{3,1}^L, b_{\text{lr}}^*(\mathbf{k})]\Omega)(L_{\text{bg}}^L - i\epsilon)^{-1}[L_{3,1}^L, b_{\text{lr}}^*(\mathbf{k})]\Omega) \\
&= \frac{1}{2} \sum_{\mathbf{p}, \mathbf{p}-\mathbf{k} \in \Xi_L^{\geq}} (V_{\mathbf{p},\mathbf{k}-\mathbf{p}} + V_{\mathbf{k}-\mathbf{p},\mathbf{p}})^2 \left(\sqrt{(1+\rho(\mathbf{p}))(1+\rho(\mathbf{p}-\mathbf{k}))(1+\rho(\mathbf{k}))} \right. \quad (2.132)
\end{aligned}$$

$$\left. - \sqrt{\rho(\mathbf{p})\rho(\mathbf{p}-\mathbf{k})\rho(\mathbf{k})} \right)^2 \frac{2i\epsilon}{(\omega_{\text{bg}}(\mathbf{k}) - \omega_{\text{bg}}(\mathbf{p}-\mathbf{k}) - \omega_{\text{bg}}(\mathbf{p}))^2 + \epsilon^2} \quad (2.133)$$

$$\begin{aligned}
& + \sum_{\mathbf{p}, \mathbf{k}-\mathbf{p} \in \Xi_L^{\geq}} (V_{-\mathbf{p},\mathbf{k}} + V_{\mathbf{k},-\mathbf{p}})^2 \left(\sqrt{(1+\rho(\mathbf{p}-\mathbf{k}))(1+\rho(\mathbf{k}))\rho(\mathbf{p})} \right. \\
& \left. - \sqrt{\rho(\mathbf{k})\rho(\mathbf{p}-\mathbf{k})(1+\rho(\mathbf{p}))} \right)^2 \frac{2i\epsilon}{(\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) - \omega_{\text{bg}}(\mathbf{p}-\mathbf{k}))^2 + \epsilon^2}, \quad (2.134)
\end{aligned}$$

where in the second line (2.132) we have symmetrized the potential

$$V_{\mathbf{p},\mathbf{k}-\mathbf{p}}^2 + V_{\mathbf{p},\mathbf{k}-\mathbf{p}}V_{\mathbf{k}-\mathbf{p},\mathbf{p}} \rightarrow \frac{1}{2}(V_{\mathbf{k}-\mathbf{p},\mathbf{p}} + V_{\mathbf{p},\mathbf{k}-\mathbf{p}})^2, \quad (2.135)$$

exploiting the symmetry under $\mathbf{p} \leftrightarrow \mathbf{k}-\mathbf{p}$ of the other factors multiplying the effective potential and of the sums $\sum_{\mathbf{p}, \mathbf{k}-\mathbf{p} \in \Xi_L^{\geq}}$. The thermal factors in (2.132), (2.134)

can be written in the following form

$$\begin{aligned} & \left(\sqrt{(1 + \rho(\mathbf{p}))(1 + \rho(\mathbf{k}))(1 + \rho(\mathbf{p} - \mathbf{k}))} - \sqrt{\rho(\mathbf{p})\rho(\mathbf{p} - \mathbf{k})\rho(\mathbf{k})} \right)^2 \\ &= \rho(\mathbf{p})\rho(\mathbf{k})\rho(\mathbf{p} - \mathbf{k}) \left(1 - e^{\beta \frac{\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p} - \mathbf{k})}{2}} \right)^2 \end{aligned} \quad (2.136)$$

$$\begin{aligned} & \left(\sqrt{(1 + \rho(\mathbf{p} - \mathbf{k}))(1 + \rho(\mathbf{k}))\rho(\mathbf{p})} - \sqrt{(1 + \rho(\mathbf{p}))\rho(\mathbf{p} - \mathbf{k})\rho(\mathbf{k})} \right)^2 \\ &= \rho(\mathbf{p})\rho(\mathbf{k})\rho(\mathbf{p} - \mathbf{k}) \left(e^{\beta \frac{\omega_{\text{bg}}(\mathbf{p})}{2}} - e^{\beta \frac{\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p} - \mathbf{k})}{2}} \right)^2. \end{aligned} \quad (2.137)$$

Hence, the final expression before taking the thermodynamic limit $L \rightarrow +\infty$ reads as

$$\begin{aligned} & -\frac{1}{2L^d} \sum_{\mathbf{p}, \mathbf{p}-\mathbf{k} \in \Xi_L^{\geq}} j(\mathbf{k}; \mathbf{p}, \mathbf{p} - \mathbf{k})^2 \rho(\mathbf{p})\rho(\mathbf{p} - \mathbf{k})\rho(\mathbf{k}) \\ & \times \left(1 - e^{\beta \frac{\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p} - \mathbf{k})}{2}} \right)^2 \frac{2i\epsilon}{(\omega_{\text{bg}}(\mathbf{k}) - \omega_{\text{bg}}(\mathbf{p} - \mathbf{k}) - \omega_{\text{bg}}(\mathbf{p}))^2 + \epsilon^2} \end{aligned} \quad (2.138)$$

$$\begin{aligned} & -\frac{1}{L^d} \sum_{\mathbf{p}, \mathbf{p}-\mathbf{k} \in \Xi_L^{\geq}} j(\mathbf{p} - \mathbf{k}; \mathbf{p}, \mathbf{k})^2 \rho(\mathbf{p})\rho(\mathbf{p} - \mathbf{k})\rho(\mathbf{k}) \\ & \times \left(e^{\beta \frac{\omega_{\text{bg}}(\mathbf{p})}{2}} - e^{\beta \frac{\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p} - \mathbf{k})}{2}} \right)^2 \frac{2i\epsilon}{(\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) - \omega_{\text{bg}}(\mathbf{p} - \mathbf{k}))^2 + \epsilon^2}. \end{aligned} \quad (2.139)$$

Since all the functions involved can be extended to continuous functions of the momenta $\mathbf{p} \in \mathbb{R}^d$, we can compute the limit $L \rightarrow +\infty$ by replacing

$$\frac{1}{L^d} \sum_{\mathbf{p}, \mathbf{p}-\mathbf{k} \in \Xi_L^{\geq}} \rightarrow \int \frac{d\mathbf{p}}{(2\pi)^d}.$$

The limit $\epsilon \searrow 0$ can then be evaluated using the limit of approximate delta functions

$$\lim_{\epsilon \searrow 0} \frac{\epsilon}{x^2 + \epsilon^2} = \pi \delta(x).$$

The well-definedness and finiteness of the integrals in (2.117) and (2.118) under our hypotheses (2.2a)–(3.5b) will be discussed in the following section (See Theorem 3.6 and Theorem 3.12). This concludes the proof. $\square$

The computation of $\delta(\mathbf{k})$ in (2.108) is analogous to that of $\xi(\mathbf{k})$, the only major difference being the absence of the $l \leftrightarrow r$ symmetry for the vector $b_l^*(\mathbf{k})\Omega$. This lack of symmetry produces additional terms contributing to $\text{Re } \delta(\mathbf{k})$.

Proposition 2.3. *Fix $\nu_0 \in]0, +\infty[$ and suppose the potential satisfies Assumptions (2.2a)–(3.5b) for all $\nu \in]0, \nu_0]$. Then, we have for the imaginary correction*

$$\text{Im} \delta(\mathbf{k}) = \frac{1}{2} \xi(\mathbf{k}), \quad (2.140)$$

This contribution is only due to the Fermi Golden Rule term. The real correction is the sum of two contributions. One coming from the Fermi Golden Rule:

$$-\frac{1}{2}\text{PV} \int \frac{d\mathbf{p}}{(2\pi)^d} \frac{j(\mathbf{k}; \mathbf{p}, \mathbf{p} - \mathbf{k})^2 \rho(\mathbf{p}) \rho(\mathbf{p} - \mathbf{k}) \rho(\mathbf{k})}{\omega_{\text{bg}}(\mathbf{p} - \mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) - \omega_{\text{bg}}(\mathbf{k})} \times \left[\left(1 - e^{\beta \frac{\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p} - \mathbf{k})}{2}} \right)^2 - \left(e^{\beta \frac{\omega_{\text{bg}}(\mathbf{k})}{2}} - e^{\beta \frac{\omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p} - \mathbf{k})}{2}} \right)^2 \right] \quad (2.141)$$

$$-\text{PV} \int \frac{d\mathbf{p}}{(2\pi)^d} \frac{j(\mathbf{p} - \mathbf{k}; \mathbf{p}, \mathbf{k})^2 \rho(\mathbf{p}) \rho(\mathbf{p} - \mathbf{k}) \rho(\mathbf{k})}{\omega_{\text{bg}}(\mathbf{p} - \mathbf{k}) - \omega_{\text{bg}}(\mathbf{p}) - \omega_{\text{bg}}(\mathbf{k})} \times \left[\left(e^{\beta \frac{\omega_{\text{bg}}(\mathbf{p})}{2}} - e^{\beta \frac{\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p} - \mathbf{k})}{2}} \right)^2 - \left(e^{\beta \frac{\omega_{\text{bg}}(\mathbf{p} - \mathbf{k})}{2}} - e^{\beta \frac{\omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{k})}{2}} \right)^2 \right] \quad (2.142)$$

$$-\frac{1}{2} \int \frac{d\mathbf{p}}{(2\pi)^d} \frac{\kappa(\mathbf{k}, \mathbf{p}, \mathbf{p} - \mathbf{k})^2 \rho(\mathbf{p}) \rho(\mathbf{p} - \mathbf{k}) \rho(\mathbf{k})}{\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p} - \mathbf{k})} \times \left[\left(1 - e^{\beta \frac{\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p} - \mathbf{k})}{2}} \right)^2 - \left(e^{\beta \frac{\omega_{\text{bg}}(\mathbf{k})}{2}} - e^{\beta \frac{\omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p} - \mathbf{k})}{2}} \right)^2 \right], \quad (2.143)$$

where $j(\mathbf{k}; \mathbf{p}, \mathbf{q})$ and $\kappa(\mathbf{k}, \mathbf{p}, \mathbf{q})$ have been defined in Equations (1.5) and (1.6), and one from the Hellmann-Feynman term:

$$\int \frac{d\mathbf{p}}{(2\pi)^d} [\hat{v}(\mathbf{0})(c_{\mathbf{k}}^2 + s_{\mathbf{k}}^2)s_{\mathbf{p}}^2 + \hat{v}(\mathbf{k} - \mathbf{p})((c_{\mathbf{k}}^2 + s_{\mathbf{k}}^2)s_{\mathbf{p}}^2 + 2c_{\mathbf{k}}s_{\mathbf{k}}c_{\mathbf{p}}s_{\mathbf{p}})] \quad (2.144)$$

$$+ \int \frac{d\mathbf{p}}{(2\pi)^d} (\hat{v}(\mathbf{0}) + \hat{v}(\mathbf{k} - \mathbf{p}))(s_{\mathbf{k}}^2 + c_{\mathbf{k}}^2)(s_{\mathbf{p}}^2 + c_{\mathbf{p}}^2)\rho(\mathbf{p}) + \int \frac{d\mathbf{p}}{(2\pi)^d} 4\hat{v}(\mathbf{k} - \mathbf{p})c_{\mathbf{k}}s_{\mathbf{k}}c_{\mathbf{p}}s_{\mathbf{p}}\rho(\mathbf{p}). \quad (2.145)$$

Proof. For the Fermi Golden Rule, the only nonzero contribution to $\delta(\mathbf{k})$ comes from vectors of the following form

$$[b_1^*(\mathbf{k}), b_1^*(\mathbf{p} + \mathbf{q})b_r^*(\mathbf{p})b_l(\mathbf{p})]\Omega = -\delta_{\mathbf{k}, \mathbf{p}}b_1^*(\mathbf{q} + \mathbf{k})b_r^*(\mathbf{p})\Omega, \quad (2.146)$$

$$[b_1^*(\mathbf{k}), b_l(\mathbf{q} + \mathbf{p})b_1^*(\mathbf{p})b_1^*(\mathbf{q})]\Omega = -\delta_{\mathbf{q} + \mathbf{p}, \mathbf{k}}b_1^*(\mathbf{k} - \mathbf{p})b_1^*(\mathbf{p})\Omega, \quad (2.147)$$

$$[b_1^*(\mathbf{k}), b_r^*(\mathbf{q} + \mathbf{p})b_r^*(-\mathbf{p})b_l(-\mathbf{q})]\Omega = -\delta_{-\mathbf{q}, \mathbf{k}}b_r^*(\mathbf{p} - \mathbf{k})b_r^*(-\mathbf{p})\Omega, \quad (2.148)$$

$$[b_l(\mathbf{k}), b_1^*(\mathbf{p} + \mathbf{q})b_r^*(\mathbf{p})b_r^*(\mathbf{p})]\Omega = \delta_{\mathbf{k}, \mathbf{p} + \mathbf{q}}b_r^*(\mathbf{p})b_r^*(\mathbf{k} - \mathbf{p})\Omega, \quad (2.149)$$

$$[b_l(\mathbf{k}), b_r^*(\mathbf{q} + \mathbf{p})b_1^*(\mathbf{p})b_1^*(\mathbf{q})]\Omega = \delta_{\mathbf{k}, \mathbf{p}}b_r^*(\mathbf{q} + \mathbf{k})b_1^*(\mathbf{q})\Omega + \delta_{\mathbf{k}, \mathbf{q}}b_r^*(\mathbf{p} + \mathbf{k})b_1^*(\mathbf{p})\Omega, \quad (2.150)$$

$$[b_l(\mathbf{k}), b_1^*(\mathbf{q} + \mathbf{p})b_1^*(-\mathbf{p})b_1^*(-\mathbf{q})]\Omega = \delta_{\mathbf{k}, -\mathbf{p}}b_1^*(\mathbf{q} - \mathbf{k})b_1^*(-\mathbf{q})\Omega + \delta_{\mathbf{k}, -\mathbf{q}}b_1^*(\mathbf{p} - \mathbf{k})b_1^*(-\mathbf{p})\Omega + \delta_{\mathbf{q} + \mathbf{p}, \mathbf{k}}b_1^*(\mathbf{p} - \mathbf{k})b_1^*(-\mathbf{p})\Omega. \quad (2.151)$$

From Equations (2.146)—(2.151) we can deduce that

$$[b_l(\mathbf{k}), [b_1^*(\mathbf{k}), L_3^L]]\Omega = 0. \quad (2.152)$$

Thus, we only need to compute

$$[b_1^*(\mathbf{k}), L_3^L]\Omega, \quad [b_l(\mathbf{k}), L_3^L]\Omega. \quad (2.153)$$

Writing L_3^L as $L_{3,1}^L + L_{3,0}^L$ we obtain

$$\begin{aligned}
& [L_{3,1}^L, b_1^*(\mathbf{k})]\Omega \\
&= \sum_{\mathbf{p}, \mathbf{p}-\mathbf{k} \in \Xi_L^{\geq}} V_{\mathbf{p}, \mathbf{k}-\mathbf{p}} \left(\sqrt{(1+\rho(\mathbf{p}))(1+\rho(\mathbf{p}-\mathbf{k}))(1+\rho(\mathbf{k}))} \right. \\
&\quad \left. - \sqrt{\rho(\mathbf{p})\rho(\mathbf{p}-\mathbf{k})\rho(\mathbf{k})} \right) b_1^*(\mathbf{p})b_1^*(\mathbf{k}-\mathbf{p})\Omega \\
&\quad + \sum_{\mathbf{p}, \mathbf{p}-\mathbf{k} \in \Xi_L^{\geq}} (V_{\mathbf{k}, -\mathbf{p}} + V_{-\mathbf{p}, \mathbf{k}}) \left(\sqrt{(1+\rho(\mathbf{p}-\mathbf{k}))(1+\rho(\mathbf{k}))\rho(\mathbf{p})} \right. \\
&\quad \left. - \sqrt{\rho(\mathbf{k})\rho(\mathbf{p}-\mathbf{k})(1+\rho(\mathbf{p}))} \right) b_1^*(\mathbf{k}-\mathbf{p})b_r^*(-\mathbf{p})\Omega,
\end{aligned} \tag{2.154}$$

$$\begin{aligned}
& - \sqrt{\rho(\mathbf{k})\rho(\mathbf{p}-\mathbf{k})(1+\rho(\mathbf{p}))} b_1^*(\mathbf{k}-\mathbf{p})b_r^*(-\mathbf{p})\Omega,
\end{aligned} \tag{2.155}$$

and

$$\begin{aligned}
& [L_{3,0}^L, b_1^*(\mathbf{k})]\Omega \\
&= \sum_{\mathbf{p}, \mathbf{p}-\mathbf{k} \in \Xi_L^{\geq}} (U_{\mathbf{k}, -\mathbf{p}} + U_{-\mathbf{p}, \mathbf{k}} + U_{\mathbf{p}, \mathbf{k}-\mathbf{p}}) \left(\sqrt{(1+\rho(\mathbf{k}))\rho(\mathbf{p})\rho(\mathbf{p}-\mathbf{k})} \right. \\
&\quad \left. - \sqrt{(1+\rho(\mathbf{p}))(1+\rho(\mathbf{p}-\mathbf{k}))\rho(\mathbf{k})} \right) b_r^*(\mathbf{p}-\mathbf{k})b_r^*(-\mathbf{p})\Omega,
\end{aligned} \tag{2.156}$$

where again we have exploited the reflection symmetry of ρ , V and U . An analogous computation reveals that the vectors $[b_l(\mathbf{k}), L_{3,1}^L]\Omega$ and $[b_l(\mathbf{k}), L_{3,0}^L]\Omega$ can be obtained respectively from $[b_1^*(\mathbf{k}), L_{3,1}^L]\Omega$ and $[b_l(\mathbf{k})^*, L_{3,0}^L]\Omega$ after exchanging the left and right labels $l \leftrightarrow r$ and permuting the thermal factors

$$\begin{aligned}
& [b_l(\mathbf{k}), L_{3,1}^L]\Omega \\
&= \sum_{\mathbf{p}, \mathbf{p}-\mathbf{k} \in \Xi_L^{\geq}} V_{\mathbf{p}, \mathbf{k}-\mathbf{p}} \left(\sqrt{(1+\rho(\mathbf{k}))\rho(\mathbf{p})\rho(\mathbf{p}-\mathbf{k})} \right. \\
&\quad \left. - \sqrt{(1+\rho(\mathbf{p}))(1+\rho(\mathbf{p}-\mathbf{k}))\rho(\mathbf{k})} \right) b_r^*(\mathbf{p})b_r^*(\mathbf{k}-\mathbf{p})\Omega \\
&\quad + \sum_{\mathbf{p}, \mathbf{p}-\mathbf{k} \in \Xi_L^{\geq}} (V_{\mathbf{k}, -\mathbf{p}} + V_{-\mathbf{p}, \mathbf{k}}) \left(\sqrt{(1+\rho(\mathbf{p}))(1+\rho(\mathbf{k}))\rho(\mathbf{p}-\mathbf{k})} \right. \\
&\quad \left. - \sqrt{\rho(\mathbf{k})\rho(\mathbf{p})(1+\rho(\mathbf{p}-\mathbf{k}))} \right) b_r^*(\mathbf{k}-\mathbf{p})b_l^*(-\mathbf{p})\Omega,
\end{aligned} \tag{2.157}$$

$$\begin{aligned}
& - \sqrt{\rho(\mathbf{k})\rho(\mathbf{p})(1+\rho(\mathbf{p}-\mathbf{k}))} b_r^*(\mathbf{k}-\mathbf{p})b_l^*(-\mathbf{p})\Omega,
\end{aligned} \tag{2.158}$$

$$\begin{aligned}
& [b_l(\mathbf{k}), L_{3,0}^L]\Omega \\
&= \sum_{\mathbf{p}, \mathbf{p}-\mathbf{k} \in \Xi_L^{\geq}} (U_{\mathbf{k}, -\mathbf{p}} + U_{-\mathbf{p}, \mathbf{k}} + U_{\mathbf{p}, \mathbf{k}-\mathbf{p}}) \left(\sqrt{(1+\rho(\mathbf{k}))(1+\rho(\mathbf{p}))(1+\rho(\mathbf{p}-\mathbf{k}))} \right. \\
&\quad \left. - \sqrt{\rho(\mathbf{k})\rho(\mathbf{p}-\mathbf{k})\rho(\mathbf{p})} \right) b_l^*(\mathbf{p}-\mathbf{k})b_l^*(-\mathbf{p})\Omega.
\end{aligned} \tag{2.159}$$

From (2.154)–(2.159) we see immediately that

$$([b_1^*(\mathbf{k}), L_{3,1}^L]\Omega | (L_{\text{bg}} - \omega_{\text{bg}}(\mathbf{k}) - i\epsilon)^{-1} [b_1^*(\mathbf{k}), L_{3,0}^L]\Omega) \tag{2.160}$$

$$= ([b_l(\mathbf{k}), L_{3,1}^L]\Omega | (L_{\text{bg}} + \omega_{\text{bg}}(\mathbf{k}) - i\epsilon)^{-1} [b_l(\mathbf{k}), L_{3,0}^L]\Omega) = 0. \tag{2.161}$$

Writing the thermal factor as in (2.136),(2.137) the final expression before taking the thermodynamic limit will be given by

$$-\frac{1}{2L^d} \sum_{\mathbf{p}, \mathbf{p}-\mathbf{k} \in \Xi_L^>} \frac{j(\mathbf{k}; \mathbf{p}, \mathbf{p}-\mathbf{k})^2 \rho(\mathbf{p}) \rho(\mathbf{p}-\mathbf{k}) \rho(\mathbf{k})}{\omega_{\text{bg}}(\mathbf{p}-\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) - \omega_{\text{bg}}(\mathbf{k}) - i\epsilon} \times \left[\left(1 - e^{\beta \frac{\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p}-\mathbf{k})}{2}} \right)^2 - \left(e^{\beta \frac{\omega_{\text{bg}}(\mathbf{k})}{2}} - e^{\beta \frac{\omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p}-\mathbf{k})}{2}} \right)^2 \right] \quad (2.162)$$

$$-\frac{1}{L^d} \sum_{\mathbf{p}, \mathbf{p}-\mathbf{k} \in \Xi_L^>} \frac{j(\mathbf{p}-\mathbf{k}; \mathbf{p}, \mathbf{k})^2 \rho(\mathbf{p}) \rho(\mathbf{p}-\mathbf{k}) \rho(\mathbf{k})}{\omega_{\text{bg}}(\mathbf{p}-\mathbf{k}) - \omega_{\text{bg}}(\mathbf{k}) - \omega_{\text{bg}}(\mathbf{p}) - i\epsilon} \times \left[\left(e^{\beta \frac{\omega_{\text{bg}}(\mathbf{p})}{2}} - e^{\beta \frac{\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}-\mathbf{k})}{2}} \right)^2 - \left(e^{\beta \frac{\omega_{\text{bg}}(\mathbf{p}-\mathbf{k})}{2}} - e^{\beta \frac{\omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{k})}{2}} \right)^2 \right] \quad (2.163)$$

$$-\frac{1}{2L^d} \sum_{\mathbf{p}, \mathbf{p}-\mathbf{k} \in \Xi_L^>} \frac{\kappa(\mathbf{k}, \mathbf{p}, \mathbf{p}-\mathbf{k})^2 \rho(\mathbf{p}) \rho(\mathbf{k}) \rho(\mathbf{p}-\mathbf{k})}{\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p}-\mathbf{k}) - i\epsilon} \times \left[\left(1 - e^{\beta \frac{\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p}-\mathbf{k})}{2}} \right)^2 - \left(e^{\beta \frac{\omega_{\text{bg}}(\mathbf{k})}{2}} - e^{\beta \frac{\omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p}-\mathbf{k})}{2}} \right)^2 \right]. \quad (2.164)$$

As $L \rightarrow +\infty$, the summation in Equations (2.162),(2.163) and (2.164) will converge to the respective integrals. In the limit $\epsilon \searrow 0$ we use the *Sochocki-Plemelj* distributional identity on the real line

$$\lim_{\epsilon \searrow 0} \frac{1}{x - i\epsilon} = \pi i \delta(x) + \mathcal{P} \frac{1}{x}.$$

The resulting imaginary component of (2.164) is 0, as $\delta(\omega_{\text{bg}}(\mathbf{k}) + \omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p}-\mathbf{k})) = 0$. In Equation (2.162), the thermal factor

$$\left(e^{\beta \frac{\omega_{\text{bg}}(\mathbf{k})}{2}} - e^{\beta \frac{\omega_{\text{bg}}(\mathbf{p}) + \omega_{\text{bg}}(\mathbf{p}-\mathbf{k})}{2}} \right)^2$$

is in the kernel of the Dirac delta distribution $\delta(\omega_{\text{bg}}(\mathbf{k}) - \omega_{\text{bg}}(\mathbf{p}-\mathbf{k}) - \omega_{\text{bg}}(\mathbf{p}))$ and thus, it will not contribute to the imaginary part. The same holds for the rightmost thermal factor inside the square brackets of Equation (2.163). Hence, we see that the resulting contributions give exactly (2.140) for the imaginary part, and (2.141),(2.142),(2.143) for the real component.

For the Hellmann-Feynman correction, we have to compute the vector

$$[b_1(\mathbf{k}), [L_4^L, b_1^*(\mathbf{k})]]\Omega \quad (2.165)$$

and evaluate its projection on Ω . The only non zero contributions come from

$$\begin{aligned} & (\Omega [b_1(\mathbf{k}), [b_{\beta,1}^*(\mathbf{k}'+\mathbf{p}) b_{\beta,1}^*(\mathbf{q}-\mathbf{k}') b_{\beta,1}(\mathbf{q}) b_{\beta,1}(\mathbf{p}) - (l \leftrightarrow r), b_1^*(\mathbf{k})]] \Omega) \\ &= \rho(\mathbf{p})(\delta_{\mathbf{q},\mathbf{k}} \delta_{\mathbf{k}',\mathbf{0}} + \delta_{\mathbf{q},\mathbf{k}} \delta_{\mathbf{k}'+\mathbf{p},\mathbf{k}}) + \rho(\mathbf{q})(\delta_{\mathbf{p},\mathbf{k}} \delta_{\mathbf{k}',\mathbf{0}} + \delta_{\mathbf{p},\mathbf{k}} \delta_{\mathbf{q},\mathbf{k}+\mathbf{k}'}), \end{aligned} \quad (2.166)$$

and

$$(\Omega [b_1(\mathbf{k}), [b_{\beta,1}^*(\mathbf{p}) b_{\beta,1}(\mathbf{p}) - (l \leftrightarrow r), b_1^*(\mathbf{k})]] \Omega) = \delta_{\mathbf{k},\mathbf{p}}. \quad (2.167)$$

This shows that to the first non-trivial order only $L_{2,1}^L$ and $L_{4,2}^L$ contribute to the Hellman-Feynmann energy correction. Using (2.166) and (2.167) and the full

expression for $L_{2,1}^L + L_{4,2}^L$, Equations (2.76),(2.74), we obtain

$$\begin{aligned} & (\Omega[b_1(\mathbf{k}), [L_4^L, b_1^*(\mathbf{k})]\Omega) = \\ &= \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} [\hat{v}(\mathbf{0})(c_{\mathbf{k}}^2 + s_{\mathbf{k}}^2)s_{\mathbf{p}}^2 + \hat{v}(\mathbf{k} - \mathbf{p})((c_{\mathbf{k}}^2 + s_{\mathbf{k}}^2)s_{\mathbf{p}}^2 + 2c_{\mathbf{k}}s_{\mathbf{k}}c_{\mathbf{p}}s_{\mathbf{p}})] \end{aligned} \quad (2.168)$$

$$\begin{aligned} &+ \frac{1}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} (\hat{v}(\mathbf{0}) + \hat{v}(\mathbf{k} - \mathbf{p}))(s_{\mathbf{k}}^2 + c_{\mathbf{k}}^2)(s_{\mathbf{p}}^2 + c_{\mathbf{p}}^2)\rho(\mathbf{p}) \\ &+ \frac{4}{L^d} \sum_{\mathbf{p} \in \Xi_L^>} \hat{v}(\mathbf{k} - \mathbf{p})c_{\mathbf{k}}s_{\mathbf{k}}c_{\mathbf{p}}s_{\mathbf{p}}\rho(\mathbf{p}). \end{aligned} \quad (2.169)$$

Taking the limit $L \rightarrow +\infty$ Equations (2.168) and (2.169) converge to the corresponding integral over $\mathbf{p} \in \mathbb{R}^d$. We note that (2.168) does not depend on the temperature, so that this term survives the zero temperature limit $\frac{1}{\beta} \rightarrow 0$, while (2.169) is proportional to $\rho(\mathbf{p})$, which goes to zero as $e^{-\beta\omega_{\text{bg}}(\mathbf{p})}$ in the aforementioned limit. We also remark how the thermal factor $\rho(\mathbf{p})$ makes (2.169) convergent, while (2.168) necessitates a cut-off over momenta, which is naturally introduced by the Fourier transform of the potential, *cf.* Assumption (2.2b). $\square$

3. COMPUTATIONS OF DECAY RATES

In this section we fix the spatial dimension $d = 3$. We will compute the integrals (1.7), (1.8) in the small momenta and low temperature limit, i.e. $\frac{|\mathbf{k}|}{\sqrt{\nu}}, \frac{1}{\beta\nu} \rightarrow 0$. The different limits of the damping will depend also on the ratio between the temperature and the momentum $\beta\sqrt{\nu}|\mathbf{k}|$.

We will obtain integrals which can be computed explicitly in terms of *polylogarithm functions* Li_n [42] of positive integer order n . The latter are defined for $|z| < 1$ by the power series

$$\text{Li}_n(z) := \sum_{k=1}^{+\infty} \frac{z^k}{k^n}, \quad (3.1)$$

which can be extended by analytic continuation with the principal branch having a cut along $[1, +\infty[$ and agreeing with Equation (3.1) when $n \geq 2$ and $|z| \leq 1$. For $n \geq 3$, we have the following special values of the polylogarithm:

$$\text{Li}_n(1) = \zeta(n), \quad \left(\frac{d}{dz} \text{Li}_n \right) (1) = \zeta(n-1), \quad (3.2)$$

where $\zeta(n)$ is the *Riemann zeta function* evaluated at n .

Our results are collected in two theorems dedicated to the analysis of Beliaev and Landau damping's integrals. In the following, we will often consider, without any further mention of this fact, functions of a generic momentum vector $\mathbf{k}$ which only depends on its euclidean norm as functions of the one dimensional variable $k := |\mathbf{k}|$.

3.1. Assumptions. Let us describe in detail the assumptions on the potential. We fix the dimension $d = 3$. Some of our hypotheses involve the effective chemical potential $\nu > 0$. These hypotheses are to be understood in the following way: fix $\nu_0 \in]0, +\infty[$, then the potential satisfies the assumptions for every $\nu \in]0, \nu_0]$.

The first set of hypotheses was stated and their meaning was discussed in Subsect. 2.1:

$$v \in L^1(\mathbb{R}^d, d\mathbf{x}), \quad v(\mathbf{x}) \in \mathbb{R}; \quad (3.3a)$$

$$\hat{v} \in L^2(\mathbb{R}^d, d\mathbf{k}); \quad (3.3b)$$

$$\hat{v}(\mathbf{0}) > 0, \quad \hat{v}(\mathbf{k}) > -\hat{v}(\mathbf{0}) \frac{|\mathbf{k}|^2}{4\nu}; \quad \nu \in]0, \nu_0]; \quad (3.3c)$$

$$v(\mathbf{x}) = v(-\mathbf{x}), \quad \text{which implies } \hat{v}(\mathbf{k}) = \hat{v}(-\mathbf{k}). \quad (3.3d)$$

To carry out the computation of the Beliaev damping in Theorem 1.1 we will require in addition the rotational invariance of v . Moreover, we will need additional regularity assumptions for $\hat{v}(\mathbf{k})$ close to $\mathbf{k} = \mathbf{0}$, both for making sense of the Dirac delta distribution $\delta(\omega_{\text{bg}}(\mathbf{k}) - \omega_{\text{bg}}(\mathbf{p}) - \omega_{\text{bg}}(\mathbf{p} - \mathbf{k}))$ that requires a differentiability hypothesis for small momenta, and for the estimates of the remainder terms in Equations (1.9), (1.10). The additional hypotheses are the following,

$$v \text{ is rotationally invariant}; \quad (3.4a)$$

$$\hat{v} \text{ is } C^5 \text{ in a neighborhood of } \mathbf{k} = \mathbf{0}; \quad (3.4b)$$

$$\frac{\nu}{\hat{v}(0)} \frac{d^2 \hat{v}}{dk^2}(0) > -\frac{1}{2}, \quad \nu \in]0, \nu_0]. \quad (3.4c)$$

Hypotheses (3.4a) and (3.4b) ensure that we can regard its Fourier transform as a function of $k := |\mathbf{k}|$ and write it in the following form

$$\hat{v}(k) = \hat{v}(0)(1 + r(k)), \quad r(k) = O(k^2) \text{ as } k \rightarrow 0.$$

Hypothesis (3.4c) is sufficient to guarantee the existence of an interval $[0, K]$, with $K > 0$, such that $p \rightarrow \omega_{\text{bg}}(p)$ is strictly convex for $p \in [0, K]$.

To evaluate the Landau damping as in Theorem 1.2, we need to make sense of the Dirac delta distribution $\delta(\omega_{\text{bg}}(\mathbf{p} - \mathbf{k}) - \omega_{\text{bg}}(\mathbf{p}) - \omega_{\text{bg}}(\mathbf{k}))$. The support of this distribution could contain regions of arbitrarily high momenta. Thus, we will need to control the derivatives of $\omega_{\text{bg}}(\mathbf{p})$, $\omega_{\text{bg}}(\mathbf{p} - \mathbf{k})$ for all values of $\mathbf{p} \in \mathbb{R}^d$. For this reason, in addition to (3.4a) and (3.4c), we need to require

$$\hat{v} \text{ is } C^1; \quad (3.5a)$$

$$\frac{k^2}{2\nu} + \frac{\hat{v}(k)}{\hat{v}(0)} + \frac{k}{2\hat{v}(0)} \frac{d\hat{v}(k)}{dk} = 0 \text{ has only a finite number of solutions}$$

$$\text{and } \liminf_{k \rightarrow +\infty} \left(\frac{k^2}{2\nu} + \frac{\hat{v}(k)}{\hat{v}(0)} + \frac{k}{2\hat{v}(0)} \frac{d\hat{v}(k)}{dk} \right) > 0, \quad \nu \in]0, \nu_0]. \quad (3.5b)$$

We remark that these hypotheses allow for $\omega_{\text{bg}}(k)$ to have a finite number of stationary points. Such stationary points typically arise in realistic dispersion relations, for instance in superfluid ^4He [1, 4, 17, 21, 37], where the excitation spectrum develops local maxima and minima. The quasiparticles associated with these extrema are known as *maxons* and *rotons*, respectively. As we will see, the contribution from second order perturbation theory is not influenced by the presence of such excitations in the small momentum regime.

Example 3.1. To show that all Assumptions (2.2a)–(3.5b) can be satisfied we present here an example of a potential satisfying them all:

$$\hat{v}(\mathbf{k}) := v e^{-\frac{|\mathbf{k}|^2}{8\nu_0}}, \quad 1 > \frac{2v}{\nu} > 0. \quad (3.6)$$

Hypotheses (2.2a) through (3.4b) are easy to verify. The last one can be proved by noting that

$$\inf_{k>0} \left(\frac{k^2}{2\nu} + e^{-\frac{k^2}{8\nu_0}} \left(1 - \frac{k^2}{8\nu_0} \right) \right) \geq 1. \quad (3.7)$$

3.2. Changes of variables. We begin by proving the following lemma

Lemma 3.2. *Fix $\nu_0 \in]0, +\infty[$ and suppose the potential satisfies (2.2a)–(2.2d) and (3.4a)–(3.4c) for all $\nu \in]0, \nu_0]$. Then, there exists a sufficiently small neighborhood U of 0 such that $k \rightarrow \omega_{\text{bg}}(k)$ is invertible and its inverse, $\omega \rightarrow p(\omega)$, is C^5 . In particular, $U \ni \omega \rightarrow \hat{v}(p(\omega))$ is in $C^5(U)$.*

Proof. The Fourier transform of the potential $\mathbf{k} \rightarrow \hat{v}(\mathbf{k})$ can be regarded as a one-dimensional function of $k := |\mathbf{k}|$, which is C^5 in the latter variable in some neighborhood of 0. Thus, we have that

$$\omega_{\text{bg}}(k) = \sqrt{\frac{k^4}{4} + \frac{\nu \hat{v}(k)}{\hat{v}(0)} k^2}$$

is again a C^5 function of the variable k for some $K > 0$ and $0 < k < K$. Then, we can consider the function of two variables

$$\mathbb{R}^2 \ni (\omega, k) \rightarrow g(\omega, k) := \omega - k \sqrt{\frac{k^2}{4} + \frac{\nu \hat{v}(k)}{\hat{v}(0)}}. \quad (3.8)$$

It is immediate to verify that

$$g(0, 0) = 0, \quad (\partial_k g)(0, 0) = -\sqrt{\nu} \neq 0, \quad (3.9)$$

so that by the implicit function theorem we can find a neighborhood U of $\omega = 0$ and a (unique) C^5 function $U \ni \omega \rightarrow p(\omega)$ such that $g(\omega, p(\omega)) = 0$. Hence, for $\omega_{\text{bg}}(\mathbf{k}) \in U$, $\hat{v}(\mathbf{k})$ can be implicitly represented as a C^5 function of the Bogoliubov energy $\omega_{\text{bg}}(\mathbf{k})$. $\square$

The previous lemma allows us to use as integration variables, at least for sufficiently small momenta, the Bogoliubov energies $u := \omega_{\text{bg}}(\mathbf{p})$, and $w := \omega_{\text{bg}}(\mathbf{q})$. For arbitrary values of the momenta, the relation $k \rightarrow \omega_{\text{bg}}(k)$ need not be globally invertible. However, if $\hat{v}$ is at least C^1 , one has

$$\frac{d\omega_{\text{bg}}(k)}{dk} = \frac{k^3}{2\omega_{\text{bg}}(k)} \left(1 + \frac{2\nu \hat{v}(k)}{\hat{v}(0)k^2} + \frac{\nu}{\hat{v}(0)k} \frac{d\hat{v}(k)}{dk} \right), \quad k > 0, \quad (3.10)$$

which by Assumptions (3.5a)–(3.5b) can only be zero at a finite number of points and is positive for sufficiently large k . This implies that there are always a finite number of intervals where $k \rightarrow \omega_{\text{bg}}(k)$ can be inverted.

For completeness, we illustrate a list of relevant changes of variables that one can perform to simplify the integrals. A similar presentation can be found in [26]. First of all, it is convenient to employ spherical coordinates to integrate out the azimuthal coordinate. This is done by choosing the z -axis parallel to $\mathbf{k}$. We can represent this change of variables as $\mathbf{p} = (p \sin \theta \cos \phi, p \sin \theta \sin \phi, p \cos \theta)$, so that the integration measure is changed as

$$d\mathbf{p} = p^2 dp d(\cos \theta) d\phi. \quad (3.11)$$

Due to the symmetry of the problem we can always perform the ϕ integration, which results in a global factor of 2π . Now, we can move to more natural coordinates for the problem given by $p := |\mathbf{p}|$ and $q := |\mathbf{p} - \mathbf{k}|$. The Jacobian of this transformation is computed to be

$$p^2 dp d(\cos \theta) = \frac{pq}{k} dp dq \quad (3.12)$$

and the integration range can be read from the constraints

$$|p - q| \leq k, \quad (3.13)$$

$$k \leq p + q, \quad (3.14)$$

that follow from the triangle inequality.

If we now restrict to intervals such that both $p \rightarrow \omega_{\text{bg}}(p)$ and $q \rightarrow \omega_{\text{bg}}(q)$ are differentiable and can be inverted, we can use the variables $u := \omega_{\text{bg}}(p)$ and $w := \omega_{\text{bg}}(q)$. After this, the measure changes as follows

$$\frac{pq}{k} dp dq = \frac{uw}{k} |f(u)f(w)| du dw, \quad (3.15)$$

with

$$f(u) := \frac{dp(u)^2}{d(u^2)} \quad (3.16)$$

$$= \left(\frac{\nu \hat{v}(p(u))}{\hat{v}(0)} + \left(\frac{1}{2} + \frac{\nu}{\hat{v}(0)} \frac{d\hat{v}}{d(p^2)}(p(u)) \right) p(u)^2 \right)^{-1} \quad (3.17)$$

where $u \rightarrow p(u)$ is the inverse of $p \rightarrow \omega_{\text{bg}}(p)$. Note that by Assumptions (2.2c), (3.4b) f is always a bounded function for u in a neighborhood of 0. Furthermore, if we assume (3.5a) – (3.5b) we see that f is well defined and bounded for sufficiently large u .

Another useful change of variables that we will employ is

$$x = u + w, \quad y = u - w, \quad (3.18)$$

$$u = \frac{x + y}{2}, \quad w = \frac{x - y}{2}, \quad (3.19)$$

which transforms the measure as

$$\frac{uw}{k} |f(u)f(w)| du dw = \frac{x^2 - y^2}{8k} \left| f\left(\frac{x+y}{2}\right) f\left(\frac{x-y}{2}\right) \right| dx dy. \quad (3.20)$$

In addition, it will be convenient to introduce the following rescalings and shifts, which allow us to factor out the dependence on the physical parameters from the integrals:

$$t := \frac{y}{\omega_{\text{bg}}(k)}, \quad (3.21)$$

for the Beliaev damping, and

$$t := \beta \nu \frac{x - \omega_{\text{bg}}(k)}{2\nu}, \quad (3.22)$$

for the Landau damping.

3.3. Estimates for remainders. In this subsection we will prove some preliminary results that will help us characterize the asymptotic properties of the function $j(\mathbf{k}; \mathbf{p}, \mathbf{q})$ appearing in the Beliaev and Landau integrals. We begin by proving a lemma that will allow us to carefully estimate the remainder terms arising in some of the subsequent expansions.

Lemma 3.3. *Consider a function of two variables $U \ni (u, w) \rightarrow R(u, w)$ defined in a neighborhood U of $(0, 0)$. Suppose that*

- i) $R \in C^3(U)$.
- ii) $R(0, w) = R(u, 0) = 0$.

Then, the following holds

$$R(u, w) = uw \int_0^1 \int_0^1 ds_1 ds_2 (\partial_u \partial_w R)(s_1 u, s_2 w) \quad (3.23)$$

Proof. The proof is two applications of the fundamental theorem of calculus. We have

$$R(u, w) = R(u, w) - R(0, w) \quad (3.24)$$

$$= u \int_0^1 ds_1 (\partial_u R)(s_1 u, w) \quad (3.25)$$

$$= uw \int_0^1 \int_0^1 ds_1 ds_2 (\partial_u \partial_w R)(s_1 u, s_2 w), \quad (3.26)$$

where in the last line (3.26) we have used that

$$0 = R(u, 0) = u \int_0^1 ds_1 (\partial_u R)(s_1 u, 0), \quad (3.27)$$

by the equality in the second line (3.25) and the second hypothesis on R . $\square$

Another result we will use is the following

Lemma 3.4. *Take integers $l, m \geq 0$ and $n \geq l + m + 1$. Consider a C^{n-l-m} function $(u, w) \rightarrow f(u, w)$ satisfying*

$$u^l w^m f(u, w) = O((u^2 + w^2)^{\frac{n}{2}}), \quad \text{as } (u, w) \rightarrow (0, 0). \quad (3.28)$$

Then

$$f(u, w) \in O((u^2 + w^2)^{\frac{n-l-m}{2}}). \quad (3.29)$$

Proof. We will prove the Lemma by induction on n . If f satisfies the hypotheses, then we have

$$|u^{l+m} f(u, u)| \leq C|u^n|, \quad \text{as } u \rightarrow 0 \quad (3.30)$$

for some constant $C > 0$. As $n > l + m$, this implies that $f(0, 0) = 0$. Then, we can write f as

$$f(u, w) = \int_0^1 ds [u(\partial_u f)(su, sw) + w(\partial_w f)(su, sw)] \quad (3.31)$$

If $n = l + m + 1$, the proof is complete. We proceed now by induction. Let us denote $n - l - m =: k$ to ease the notation. Suppose that the statement has been proved for $n - 1 \geq l + m + 1$. Then, if f satisfies the hypotheses of the Lemma with integer n , we get that

$$u^l w^m f(u, w) \in O((u^2 + w^2)^{\frac{n}{2}}) \subset O((u^2 + w^2)^{\frac{n-1}{2}}). \quad (3.32)$$

Thus, by induction f satisfies $f(u, w) \in O((u^2 + w^2)^{\frac{k-1}{2}})$. Hence, by the differentiability properties of f , we know that

$$(\partial_u^i \partial_w^j f)(0, 0) = 0, \quad \text{for all } i, j \in \mathbb{Z}_+, i + j \leq k - 2. \quad (3.33)$$

Iterating the expansion in (3.31) for the derivatives of f , we see that f can be written as

$$\begin{aligned} f(u, w) &= \int_0^1 ds_1 \cdots \int_0^1 ds_{k-1} \\ &\times \sum_{i=0}^{k-1} \binom{k-1}{i} u^i w^{k-1-i} (\partial_u^i \partial_w^{k-1-i} f)(s_1 \dots s_{k-1} u, s_1 \dots s_{k-1} w). \end{aligned} \quad (3.34)$$

Let us now take $w = \alpha u$, for some arbitrary $\alpha \in \mathbb{R} \setminus \{0\}$. we have that

$$|\alpha|^m f(u, \alpha u) \leq C|u|^k, \quad (3.35)$$

with $k > 0$. This last relation implies that

$$P(\alpha) := \sum_{i=0}^{k-1} \alpha^{k-1-i} \binom{k-1}{i} (\partial_u^i \partial_w^{k-1-i} f)(0, 0) = 0, \quad (3.36)$$

for every $\alpha \in \mathbb{R} \setminus \{0\}$, and thus, by continuity, for every $\alpha \in \mathbb{R}$. Now, $\alpha \rightarrow P(\alpha)$ is equal to the zero polynomial in α . By linear independence of the monomials in α , all of P 's coefficients must be zero, i.e.

$$(\partial_u^i \partial_w^{k-1-i} f)(0, 0) = 0. \quad (3.37)$$

recalling that $f \in C^k = C^{n-l-m}$, we can iterate once again (3.31) to obtain

$$\begin{aligned} f(u, w) &= \int_0^1 ds_1 \cdots \int_0^1 ds_k \\ &\times \sum_{i=0}^k \binom{k}{i} u^i w^{k-i} (\partial_u^i \partial_w^{k-i} f)(s_1 \dots s_k u, s_1 \dots s_k w), \end{aligned} \quad (3.38)$$

which concludes the proof. $\square$

The previous lemma can be easily generalized to any finite number of variables. Let us now consider the function $(\mathbf{k}, \mathbf{p}, \mathbf{q}) \rightarrow j(\mathbf{k}; \mathbf{p}, \mathbf{q})$, which appears in the Beliaev and Landau damping integrals (2.117) and (2.118). The next proposition contains the main properties of some auxiliary functions which will appear inside the integrals after changing variables. We will denote by $u \rightarrow p(u)$ the inverse function of $p \rightarrow \omega_{\text{bg}}(p)$ in a sufficiently small neighborhood of 0, see Lemma 3.2.

Proposition 3.5. *Fix a $\nu_0 \in]0, +\infty[$ and suppose the potential satisfies (2.2a)–(2.2d) and (3.4a)–(3.4c) for any $\nu \in]0, \nu_0]$. Consider the function*

$$F(\omega; u, w) := \sqrt{8\omega u w} \sqrt{\frac{\nu}{\hat{v}(0)}} j(p(\omega); p(u), p(w)), \quad (3.39)$$

which is well defined in a sufficiently small neighborhood of $(0, 0, 0)$. Then, F satisfies the following properties

- i) It is invariant under the exchange of the last two variables $F(\omega; u, w) = F(\omega; w, u)$;
- ii) F is at least C^5 in a sufficiently small neighborhood of $(0, 0, 0)$ and it satisfies

$$F(\omega; 0, \omega) = F(\omega; \omega, 0) = 0 \quad (3.40)$$

for every sufficiently small $\omega \geq 0$;

Moreover, $G(u, w) := F(u + w; u, w)^2$ satisfies the following expansion

iii)

$$\frac{1}{\nu^5} G(u, w) = \frac{9}{\nu^6} u^2 w^2 (u + w)^2 + \frac{u^2 w^2}{\nu^4} S(u, w) \quad (3.41)$$

where $(u, w) \rightarrow S(u, w)$ is a continuous functions of order $O(\frac{u^4 + w^4}{\nu^4})$ as $(u, w) \rightarrow (0, 0)$.

Proof. i) We start by defining the following auxiliary functions of the energy

$$\nu(\omega) := \nu \frac{\hat{v}(p(\omega))}{\hat{v}(0)} \quad (3.42)$$

$$c(\omega) := \sqrt{\omega^2 + \nu(\omega)^2} + \omega \quad (3.43)$$

$$s(\omega) := \sqrt{\omega^2 + \nu(\omega)^2} - \omega. \quad (3.44)$$

Equations (3.43) and (3.44) are just regularized versions of $c_{\mathbf{k}}$ and $s_{\mathbf{k}}$ obtained by multiplying by $\sqrt{2\omega_{\text{bg}}(\mathbf{k})}$ and expressing everything in terms of the energies. Then, we have $\nu(0) = \nu$ and $c(0) = s(0) = \sqrt{\nu}$. We can write F in terms of these as

$$\begin{aligned} F(\omega; u, w) = & \nu(\omega)(s(\omega) - c(\omega))(c(u)s(w) + c(w)s(u)) \\ & + \nu(u)(c(u) - s(u))(c(\omega)c(w) + s(\omega)s(w)) \\ & + \nu(w)(c(w) - s(w))(c(\omega)c(u) + s(\omega)s(u)). \end{aligned} \quad (3.45)$$

From Equation (3.45) we read immediately the symmetry of F under $u \leftrightarrow w$. *ii)* As a consequence of Lemma 3.2, $\omega \rightarrow \nu(\omega)$ and $\omega \rightarrow p(\omega)$ are C^5 functions in a neighborhood of 0. This immediately implies that s and c are C^5 and thus, also F is of the same class of differentiability close to the origin. To verify the identity (3.40) it is sufficient to observe that $c(0) - s(0) = 0$ which implies $F(\omega; u, 0) = -F(u; \omega, 0)$. *iii)* As G is the square of F , it is of class at least C^5 . Hence, we can take derivatives of G . Its first-order derivatives will be of the form F times first order derivatives of F and thus, we have by *ii)*

$$(\partial_u G)(u, 0) = \partial_u G(0, w) = (\partial_w G)(u, 0) = \partial_w G(0, w) = 0. \quad (3.46)$$

We proceed now to perform an expansion of G as $(u, w) \rightarrow (0, 0)$. Let us write $\hat{v}$ as

$$\hat{v}(p) = \hat{v}(0)(1 + r(p)), \quad r(p) = O(p^2) \text{ as } p \rightarrow 0. \quad (3.47)$$

Then, we will have

$$\nu(\omega)^2 = \nu^2 + 2\nu^2 r(p(\omega)) + \nu^2 r(p(\omega))^2. \quad (3.48)$$

With a slight abuse of notation we will denote $r(\omega) \equiv r(p(\omega))$. Thus, we have the expansions

$$c(\omega) - s(\omega) = \sqrt{\nu} \left(\frac{\omega}{\nu} - \frac{\omega r(\omega)}{2\nu} - \frac{\omega^3}{8\nu^3} + R_1(\omega) \right), \quad (3.49)$$

$$\begin{aligned} & c(u)s(w) + c(w)s(u) \\ &= \nu \left(2 + r(u) + r(w) + \frac{1}{4\nu^2}(u - w)^2 + R_2(u, w) \right), \end{aligned} \quad (3.50)$$

$$\begin{aligned} & c(u)c(w) + s(u)s(w) \\ &= \nu \left(2 + r(u) + r(w) + \frac{1}{4\nu^2}(u + w)^2 + R_3(u, w) \right), \end{aligned} \quad (3.51)$$

where R_1 and R_2, R_3 are suitable remainders of order $O\left(\frac{\omega^5}{\nu^5}\right)$, $O\left(\frac{u^4 + w^4}{\nu^4}\right)$ respectively, obtained from the Taylor expansion in ω, u, w and r . Inserting these expansions in Equation (3.45), we can expand F as

$$F(u + w; u, w) = \frac{3}{\sqrt{\nu}} u w (u + w) + \nu^{5/2} R(u, w) \quad (3.52)$$

where R is some remainder of order $O\left(\frac{u^5 + w^5}{\nu^5}\right)$. We can also write R in terms of F as

$$R(u, w) = \frac{1}{\nu^{5/2}} \left(F(u + w; u, w) - \frac{3}{\sqrt{\nu}} u w (u + w) \right). \quad (3.53)$$

Thanks to the properties of the function appearing on the right, we deduce that R is C^5 , symmetric under the exchange $u \leftrightarrow w$ and it satisfies

$$R(u, 0) = R(0, w) = 0. \quad (3.54)$$

By Lemma 3.3 we know that $R(u, w)$ can be written as

$$R(u, w) = u w \int_0^1 \int_0^1 ds_1 ds_2 (\partial_u \partial_w R)(s_1 u, s_2 w) = O\left(\frac{u^5 + w^5}{\nu^5}\right), \quad (3.55)$$

and by Lemma 3.4 we know that the integral function in (3.55) satisfies

$$\int_0^1 \int_0^1 ds_1 ds_2 (\partial_u \partial_w R)(s_1 u, s_2 w) = O\left(\frac{u^3 + w^3}{\nu^5}\right) \quad (3.56)$$

After taking the square of $F(u + w; u, w)$ we obtain

$$\begin{aligned} \frac{1}{\nu^5} G(u, w) &= \frac{9}{\nu^6} u^2 w^2 (u + w)^2 \\ &\quad + \frac{6}{\nu^3} u w (u + w) R(u, w) \end{aligned} \quad (3.57)$$

$$+ R^2(u, w). \quad (3.58)$$

After writing R as in (3.55) we get that both the term in (3.57) and in (3.58) are of the correct form (3.41). $\square$

3.4. Beliaev damping rate. We can now start with the computation of the Beliaev damping rate:

Theorem 3.6. *Fix a $\nu_0 \in]0, +\infty[$ and suppose the potential satisfies (2.2a)–(2.2d) and (3.4a)–(3.4c) for any $\nu \in]0, \nu_0]$. Then:*

(a) *the Beliaev damping rate is computed to be*

$$\gamma_B(\mathbf{k}; \beta, \nu) = \frac{9\hat{v}(\mathbf{0})\nu^{3/2}}{2048\pi} \frac{|\mathbf{k}|^4}{\nu^2} \frac{1}{\beta\nu} \mathcal{I}(\beta\omega_{\text{bg}}(\mathbf{k})) \left(1 + O\left(\frac{|\mathbf{k}|^2}{\nu}\right)\right) \quad \text{as } \frac{|\mathbf{k}|}{\sqrt{\nu}} \rightarrow 0, \quad (3.59)$$

where

$$\mathcal{I}(\theta) = \theta(1 - e^{-\theta}) \int_{-1}^1 dt \frac{(1 - t^2)^2}{(1 - e^{-\theta\frac{1+t}{2}})(1 - e^{-\theta\frac{1-t}{2}})} \quad (3.60)$$

is a continuous function of $\theta \in [0, +\infty)$ satisfying

$$\mathcal{I}(\theta) = \frac{16}{3} + O(\theta) \quad \text{as } \theta \rightarrow 0$$

$$\mathcal{I}(\theta) = \frac{16}{15}\theta + O\left(\frac{1}{\theta^2}\right) \quad \text{as } \theta \rightarrow +\infty$$

(b) *In the small temperature regime, $\frac{1}{\beta\sqrt{\nu}|\mathbf{k}|} \rightarrow 0$, the Beliaev damping rate can be estimated as*

$$\begin{aligned} \gamma_B(\mathbf{k}, \beta, \nu) &= \frac{3\hat{v}(\mathbf{0})\nu^{3/2}}{640\pi} \frac{|\mathbf{k}|^5}{\nu^{5/2}} \left(1 + O\left(\frac{1}{(\beta\sqrt{\nu}|\mathbf{k}|)^3} + \frac{|\mathbf{k}|^2}{\nu}\right)\right) \\ &\quad \text{as } \frac{|\mathbf{k}|}{\sqrt{\nu}}, \frac{1}{\beta\sqrt{\nu}|\mathbf{k}|} \rightarrow 0. \end{aligned} \quad (3.61)$$

(c) *In the opposite limit $\beta\sqrt{\nu}|\mathbf{k}| \rightarrow 0$, the Beliaev damping rate can be estimated as*

$$\begin{aligned} \gamma_B(\mathbf{k}; \beta, \nu) &= \frac{3\hat{v}(\mathbf{0})\nu^{3/2}}{128\pi} \frac{|\mathbf{k}|^4}{\nu^2} \frac{1}{\beta\nu} \left(1 + O\left(\beta\sqrt{\nu}|\mathbf{k}| + \frac{|\mathbf{k}|^2}{\nu}\right)\right) \\ &\quad \text{as } \frac{|\mathbf{k}|}{\sqrt{\nu}}, \beta\sqrt{\nu}|\mathbf{k}| \rightarrow 0. \end{aligned} \quad (3.62)$$

Remark 3.7. We observe that Equations (3.61) and (3.62) are independent of the momentum dependent term $r(\mathbf{k})$ of the interaction potential. This implies that, to the leading order in the small-momentum limit, the process depends on the interaction solely through the value of its Fourier transform at zero, $\hat{v}(\mathbf{0})$.

Proof. (a) For sufficiently small $\mathbf{k}$, the support of the Dirac delta distribution

$$\delta(\omega_{\text{bg}}(\mathbf{k}) - \omega_{\text{bg}}(\mathbf{p}) - \omega_{\text{bg}}(\mathbf{p} - \mathbf{k})),$$

will be restricted to a neighborhood of $p := |\mathbf{p}| = 0$

$$U_{\mathbf{k}} \subset \{\mathbf{p} \in \mathbb{R}^3 | p \in [0, k]\},$$

which can be taken as small as we like for $k \rightarrow 0$. Thus, by Lemma 3.2 we can always suppose that the set $U_{\mathbf{k}}$ is contained in a neighborhood of 0, where the Bogoliubov dispersion relations $p \rightarrow \omega_{\text{bg}}(p)$ and $|\mathbf{p} - \mathbf{k}| = q \rightarrow \omega_{\text{bg}}(q)$ can be inverted. Thus, we can change variables as in Equation (3.15) taking $u = \omega_{\text{bg}}(p)$ and $w = \omega_{\text{bg}}(q)$. Hence, (1.7) takes the form

$$\begin{aligned} & \frac{\hat{v}(0)}{64\pi\nu k\omega_{\text{bg}}(k)} \int du dw G(u, w) f(u) f(w) \\ & \times \frac{(1 - e^{\frac{\beta}{2}(\omega_{\text{bg}}(k) + u + w)})^2}{(e^{\beta\omega_{\text{bg}}(\mathbf{k})} - 1)(e^{\beta u} - 1)(e^{\beta w} - 1)} \delta(\omega_{\text{bg}}(k) - u - w) \end{aligned} \quad (3.63)$$

where $G(u, w)$ has been defined in Proposition 3.5. Following Proposition 3.5 we can now write

$$G(u, w) = \frac{9}{\nu} u^2 w^2 (u + w)^2 + \nu^5 \frac{u^2 w^2}{\nu^4} S(u, w), \quad (3.64)$$

where $S(u, w)$ is a remainder of order $O\left(\frac{u^4 + w^4}{\nu^4}\right)$ as in Equations (3.57), (3.58). In particular, since $u \rightarrow f(u)$ is a bounded, continuous function in the neighborhood $U_{\mathbf{k}}$, the expression in (3.41) guarantees that the integral of the remainder is convergent also when the energies u and w approach 0 and the thermal factor in (3.63) becomes divergent. To evaluate the Dirac delta distribution we change now variables as in Equations (3.18) and (3.19), so that $\delta(\omega_{\text{bg}}(k) - u - w) = \delta(\omega_{\text{bg}}(k) - x)$. If we carry out the x integration, we see that the remaining integral in y has an integration domain given by $-\omega_{\text{bg}}(k) \leq y \leq \omega_{\text{bg}}(k)$. This is a consequence of the convexity Assumption (3.4c) and of [26, Lemma 4].

Let us first focus on the leading term on the right-hand side of (3.64); the remainder S will be treated afterward.

We have to compute

$$\begin{aligned} & \frac{9\hat{v}(0)\omega_{\text{bg}}(k)}{2048\pi\nu^2 k} (e^{\beta\omega_{\text{bg}}(k)} - 1) \int_{-\omega_{\text{bg}}(k)}^{\omega_{\text{bg}}(k)} dy (\omega_{\text{bg}}(k)^2 - y^2)^2 \\ & \times f\left(\frac{\omega_{\text{bg}}(k) + y}{2}\right) f\left(\frac{\omega_{\text{bg}}(k) - y}{2}\right) \frac{1}{(e^{\beta\frac{\omega_{\text{bg}}(k)+y}{2}} - 1)(e^{\beta\frac{\omega_{\text{bg}}(k)-y}{2}} - 1)}. \end{aligned} \quad (3.65)$$

Now, f is a bounded, sufficiently regular function which can be expanded around $u = 0$ as $f(u) = \nu^{-1} \left(+O\left(\frac{u^2}{\nu^2}\right) \right)$. Thus, after further rescaling the integration variable as $y := t\omega_{\text{bg}}(k)$ and denoting $\theta := \beta\omega_{\text{bg}}(k)$ we obtain

$$\begin{aligned} & \frac{9\hat{v}(0)\omega_{\text{bg}}(k)^5}{2048\pi\nu^4 k} \frac{1}{\beta\nu} \theta (1 - e^{-\theta}) \int_{-1}^1 dt (1 - t^2)^2 \\ & \times \frac{1}{(1 - e^{-\frac{\theta(1-t)}{2}})(1 - e^{-\frac{\theta(1+t)}{2}})} \left(1 + O\left(\frac{\omega_{\text{bg}}(k)^2}{\nu^2}\right) \right), \end{aligned} \quad (3.66)$$

which gives exactly Equation (3.59) after expanding the Bogoliubov dispersion relation $\omega_{\text{bg}}(k) = \sqrt{\nu}k + O\left(\frac{k^3}{\sqrt{\nu}}\right)$.

It only remains to examine the remainder $S(u, w)$ in (3.64). After changing variables as we already did for the main term, we are left with

$$\begin{aligned} & \frac{\hat{v}(0)\nu^2}{128\pi k} \frac{\omega_{\text{bg}}(k)^4}{\nu^4} (1 - e^{-\theta}) \int_{-1}^1 dt (1 - t^2)^2 S\left(\omega_{\text{bg}}(k) \frac{1+t}{2}, \omega_{\text{bg}}(k) \frac{1-t}{2}\right) \\ & \quad \times \frac{1}{(1 - e^{-\frac{\theta(1+t)}{2}})(1 - e^{-\frac{\theta(1-t)}{2}})} \left(1 + O\left(\frac{\omega_{\text{bg}}(k)^2}{\nu^2}\right)\right), \end{aligned} \quad (3.67)$$

where we have already expanded the functions f . Now, as $t \in [-1, 1]$ we can estimate from above

$$\left| S\left(\omega_{\text{bg}}(k) \frac{1+t}{2}, \omega_{\text{bg}}(k) \frac{1-t}{2}\right) \right| \leq K \frac{\omega_{\text{bg}}(k)^4}{\nu^4} \quad (3.68)$$

for some suitable constant $K > 0$. Thus, we estimate the integral of the remainder as

$$|(3.67)| \leq M \hat{v}(0) \nu^{3/2} \frac{\omega_{\text{bg}}(k)^7}{\nu^7} \frac{1}{\beta \nu} \theta (1 - e^{-\theta}) \int_1^1 dt \frac{(1 - t^2)^2}{(1 - e^{-\theta \frac{1+t}{2}})(1 - e^{-\theta \frac{1-t}{2}})}, \quad (3.69)$$

where M is some other positive constant. This concludes point (a).

To prove (b), we have to compute the limit of $\mathcal{I}(\theta)$ as $\theta \rightarrow +\infty$. This function can be computed explicitly in terms of polylogarithm functions of integer order. The computation goes as follows:

$$\begin{aligned} & \frac{1}{32\theta(1 - e^{-\theta})} \mathcal{I}(\theta) = \int_0^1 dt \frac{t^4 + t^2 - 2t^3}{(1 - e^{-\theta t})(1 - e^{-\theta(1-t)})} \\ & = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{-m\theta} \int_0^1 dt (t^4 + t^2 - t^3) e^{-(n-m)\theta t} \\ & = \frac{1}{30} + 4 \left(\frac{\zeta(3)}{\theta^3} - 6 \frac{\zeta(4)}{\theta^4} \right. \\ & \quad \left. + 12 \frac{\zeta(5)}{\theta^5} - \frac{\text{Li}_3(e^{-\theta})}{\theta^3} - 6 \frac{\text{Li}_4(e^{-\theta})}{\theta^4} - 12 \frac{\text{Li}_5(e^{-\theta})}{\theta^5} \right). \end{aligned} \quad (3.70)$$

To conclude one has

$$\begin{aligned} \mathcal{I}(\theta) := & 32\theta(1 - e^{-\theta}) \left[\frac{1}{30} + 4 \left(\frac{\zeta(3)}{\theta^3} - 6 \frac{\zeta(4)}{\theta^4} + 12 \frac{\zeta(5)}{\theta^5} - \frac{\text{Li}_3(e^{-\theta})}{(\theta)^3} \right. \right. \\ & \left. \left. - 6 \frac{\text{Li}_4(e^{-\theta})}{\theta^4} - 12 \frac{\text{Li}_5(e^{-\theta})}{\theta^5} \right) \right], \quad \theta \geq 0. \end{aligned} \quad (3.71)$$

Since $\text{Li}_n(e^{-\theta})$ decays exponentially as $\theta \rightarrow +\infty$, we have the asymptotic value

$$\mathcal{I}(\theta) = \frac{16}{15} \theta + O\left(\frac{1}{\theta^2}\right), \quad \text{as } \theta \rightarrow +\infty \quad (3.72)$$

which gives the correct estimate in (b). Similarly for (c) we just have to estimate $\mathcal{I}(\theta)$ for $\theta \rightarrow 0$, that is

$$\begin{aligned} \mathcal{I}(\theta) & = 32\theta(1 - e^{-\theta}) \int_0^1 dt \frac{t^2(1-t)^2}{(1 - e^{-t\theta})(1 - e^{-(1-t)\theta})} \\ & = 32 \int_0^1 dt t(1-t) + O(\theta) \\ & = \frac{16}{3} + O(\theta). \end{aligned} \quad (3.73)$$

□

3.5. Landau damping rate. In order to carry out the analysis of Landau damping, we will rely on the following Lemma

Lemma 3.8. *Let*

$$[0, +\infty[\times [0, +\infty[\ni (s, \delta) \rightarrow f(s, \delta) \quad (3.74)$$

be continuous and polinomially bounded, such that there exists a positive constant C and an integer $n \geq 0$ for which

$$|f(s, \delta)| \leq C(s + \delta)^n, \quad \text{as } (s, \delta) \rightarrow (0, 0). \quad (3.75)$$

Then, there exist strictly positive constants A, B, D , such that

$$\left| \int_0^{+\infty} dt \frac{t^2 f\left(\frac{t}{x}, \delta\right)}{\sinh\left(\frac{t}{2}\right) \sinh\left(\frac{t+\theta}{2}\right)} \right| \leq D \int_0^{+\infty} dt \frac{t^2 \left(\frac{t}{x} + \delta\right)^n}{\sinh\left(\frac{t}{2}\right) \sinh\left(\frac{t+\theta}{2}\right)} \quad (3.76)$$

uniformly for $x \geq B > 0$, $\delta \leq A$ and $\theta \geq 0$.

Proof. We separate the integral into two regions

$$\int_0^{+\infty} dt = \int_0^x dt + \int_x^{+\infty} dt. \quad (3.77)$$

Integral for $t \geq x$. We estimate the second integral first. Note that for $t \geq x > 0$, we can always find a positive constant M such that

$$\frac{1}{\sinh\left(\frac{t}{2}\right)} \leq M \frac{1}{\sinh\left(\frac{t}{4}\right)^2}, \quad (3.78)$$

where M can be chosen independently from x as $t \geq x \geq B$. By hypotheses, $f(t, \delta)$ is polynomially bounded, thus

$$\sup_{t \geq x, \delta \in [0, A]} \frac{|f\left(\frac{t}{x}, \delta\right)|}{\sinh\left(\frac{t}{4}\right)} =: L < +\infty \quad (3.79)$$

Hence, we can estimate the integral as

$$\left| \int_x^{+\infty} dt \frac{t^2 f\left(\frac{t}{x}, \delta\right)}{\sinh\left(\frac{t}{2}\right) \sinh\left(\frac{t+\theta}{2}\right)} \right| \leq ML \int_x^{+\infty} dt \frac{t^2 \left(\frac{t}{x} + \delta\right)^n}{\sinh\left(\frac{t}{4}\right) \sinh\left(\frac{t+\theta}{2}\right)} \quad (3.80)$$

$$\leq 2ML \int_{\frac{x}{2}}^{+\infty} dt \frac{t^2 \left(\frac{t}{x} + \delta\right)^n}{\sinh\left(\frac{t}{2}\right) \sinh\left(\frac{t+\theta}{2}\right)} \quad (3.81)$$

$$\leq 2ML \int_0^{+\infty} dt \frac{t^2 \left(\frac{t}{x} + \delta\right)^n}{\sinh\left(\frac{t}{2}\right) \sinh\left(\frac{t+\theta}{2}\right)}, \quad (3.82)$$

where we also used that $\frac{t}{x} + \delta \geq 1$ for $\delta \geq 0$, $t \geq x$.

Integral for $0 \leq t \leq x$. By hypotheses, there exists a neighborhood U of $(0, 0)$ in $[0, +\infty]^2$ such that

$$\left| f\left(\frac{t}{x}, \delta\right) \right| \leq C \left(\frac{t}{x} + \delta\right)^n, \quad \text{for } \left(\frac{t}{x}, \delta\right) \in U. \quad (3.83)$$

We fix now

$$0 < \frac{1}{C'} := \min_{(s, \delta) \in U^c \cap [0, 1] \times [0, A]} (s + \delta)^n, \quad (3.84)$$

and

$$F := \max_{(s, \delta) \in [0, 1] \times [0, A]} |f(s, \delta)| < +\infty. \quad (3.85)$$

Then, we will have

$$\left| f\left(\frac{t}{x}, \delta\right) \right| \leq \max\{C, C'F\} \left(\frac{t}{x} + \delta\right)^n. \quad (3.86)$$

Inserting the latter inequality in the integral for $t \leq x$ concludes the proof. $\square$

An important step in estimating (1.8) is to determine the integration domain remaining after evaluating the Dirac delta distribution. Writing $p = |\mathbf{p}|$, $q = |\mathbf{p} - \mathbf{k}|$, and $k = |\mathbf{k}|$, the relevant resonance condition is

$$y = \omega_{\text{bg}}(p) - \omega_{\text{bg}}(q) = -\omega_{\text{bg}}(k).$$

The following two lemmas describe the admissible radii for sufficiently small k and exclude resonances near stationary points of the dispersion relation.

Throughout the argument, $\nu \in]0, \nu_0]$ is fixed. The constants may depend on ν and the potential, but are independent of k and β . We require the dispersion relation to be real and strictly positive for $p > 0$; equivalently,

$$\hat{v}(p) > -\frac{\hat{v}(0)p^2}{4\nu}, \quad p > 0. \quad (3.87)$$

In particular, this condition holds if $\hat{v} \geq 0$.

Lemma 3.9. *Fix $\nu_0 > 0$ and suppose that the potential satisfies (2.2a)–(2.2d) and (3.4a)–(3.4c) for every $\nu \in]0, \nu_0]$. For each fixed $\nu \in]0, \nu_0]$, there exist constants $K, \delta > 0$ such that, for every $\mathbf{k} \in \mathbb{R}^3$ with $0 < k = |\mathbf{k}| \leq \delta$ and every $p \in [0, K]$, there is a vector $\mathbf{p} \in \mathbb{R}^3$ satisfying $|\mathbf{p}| = p$ and*

$$\omega_{\text{bg}}(\mathbf{p}) - \omega_{\text{bg}}(\mathbf{p} - \mathbf{k}) = -\omega_{\text{bg}}(\mathbf{k}). \quad (3.88)$$

Proof. The assumptions on the potential imply that ω_{bg} is C^5 , strictly increasing, and strictly convex on $[0, \rho]$, for some $\rho > 0$. Choose $K, \delta > 0$ so that $\delta < K$ and $K + \delta < \rho$. The case $p = 0$ is immediate. Suppose therefore that $p > 0$.

If $\mathbf{p}$ is parallel to $\mathbf{k}$ and points in the same direction, then $q = |p - k|$ and

$$\omega_{\text{bg}}(p) - \omega_{\text{bg}}(|p - k|) > -\omega_{\text{bg}}(k).$$

Indeed, the left-hand side is nonnegative if $p \geq k$; if $0 < p < k$, the inequality follows from strict monotonicity and $\omega_{\text{bg}}(p) > 0$.

If $\mathbf{p}$ points in the opposite direction, then $q = p + k$. Set

$$f_k(p) := \omega_{\text{bg}}(p) - \omega_{\text{bg}}(p + k).$$

Strict convexity gives

$$f'_k(p) = \omega'_{\text{bg}}(p) - \omega'_{\text{bg}}(p + k) < 0,$$

and hence

$$f_k(p) < f_k(0) = -\omega_{\text{bg}}(k), \quad p > 0.$$

The quantity $\omega_{\text{bg}}(p) - \omega_{\text{bg}}(|\mathbf{p} - \mathbf{k}|)$ depends continuously on the angle between $\mathbf{p}$ and $\mathbf{k}$. The intermediate value theorem therefore yields the required vector. For $p > 0$, the resonant angle lies strictly between 0 and π . $\square$

The dispersion relation need not be globally invertible, so changes to energy variables must be performed separately on its monotonicity intervals. Lemma 3.2 provides a local inverse near $p = 0$, while Lemma 3.9 shows that every sufficiently small radius admits a resonant direction. The next lemma excludes resonances whenever the derivative of the dispersion relation becomes too small between the two radii.

Lemma 3.10. *Assume the hypotheses of Lemma 3.9, (3.5a), and (3.87). For every compact set $P \subset [0, +\infty[$, there exist constants $\delta, \epsilon > 0$ such that, whenever*

$$0 < k \leq \delta, \quad p \in P, \quad q \geq 0, \quad |p - q| \leq k,$$

the condition

$$\min_{r \in [\min\{p, q\}, \max\{p, q\}]} |\omega'_{\text{bg}}(r)| \leq \epsilon \sqrt{\nu}$$

implies

$$|\omega_{\text{bg}}(p) - \omega_{\text{bg}}(q)| < \omega_{\text{bg}}(k). \quad (3.89)$$

In particular, the resonance condition cannot hold if either endpoint has derivative of absolute value at most $\epsilon \sqrt{\nu}$, or if a stationary point lies between p and q .

Proof. Choose $R > 0$ such that $P \subset [0, R]$ and restrict to $\delta \leq 1$. Then p , q , and the interval between them belong to $[0, R + 1]$. The assumptions imply that ω'_{bg} is continuous on this compact interval, with the right derivative $\omega'_{\text{bg}}(0) = \sqrt{\nu}$ at zero. By uniform continuity, we can choose $\delta > 0$ sufficiently small that

$$|\omega'_{\text{bg}}(r) - \omega'_{\text{bg}}(s)| < \frac{\sqrt{\nu}}{4} \quad \text{whenever } r, s \in [0, R + 1], \quad |r - s| \leq \delta.$$

Moreover, $\frac{\omega_{\text{bg}}(k)}{k} \rightarrow \sqrt{\nu}$ as $k \rightarrow 0$. After decreasing δ if necessary, we therefore have

$$\omega_{\text{bg}}(k) \geq \frac{3\sqrt{\nu}}{4}k, \quad 0 < k \leq \delta.$$

Take $\epsilon = \frac{1}{4}$. If the stated minimum is at most $\epsilon \sqrt{\nu}$, then $|\omega'_{\text{bg}}(r)| < \frac{\sqrt{\nu}}{2}$ throughout the interval between p and q . Thus

$$\begin{aligned} |\omega_{\text{bg}}(p) - \omega_{\text{bg}}(q)| &= \left| (p - q) \int_0^1 ds \omega'_{\text{bg}}(q + s(p - q)) \right| \\ &\leq \frac{\sqrt{\nu}}{2} |p - q| \leq \frac{\sqrt{\nu}}{2} k < \omega_{\text{bg}}(k). \end{aligned}$$

□

These two lemmas yield the following representation of the Landau damping integral.

Proposition 3.11. *Assume the hypotheses of Lemma 3.10 and (3.5b). For each fixed $\nu \in]0, \nu_0]$, there exist $\delta > 0$ and a finite integer $N \geq 0$, independent of k and β , with the following properties.*

Let $I_0, \dots, I_N$ be the intervals obtained by removing the stationary points of ω_{bg} from $[0, +\infty[$, ordered by increasing momentum, with $0 \in I_0$. Write p_n for the inverse of $\omega_{\text{bg}}|_{I_n}$ and set

$$f_n(u) := \frac{p_n(u)}{u} p'_n(u), \quad u > 0,$$

with $f_0(0) := \frac{1}{\nu}$ by continuous extension. For $s > 0$ sufficiently small and $u > 0$ with $u, u + s \in \omega_{\text{bg}}(I_n)$, define

$$G_n(u, s) := \frac{8\nu}{\hat{v}(0)} us(u + s) j(p_n(u + s); p_0(s), p_n(u))^2. \quad (3.90)$$

Here j is regarded as a function of the three momentum magnitudes, using rotational invariance. On the branch containing zero, G_0 agrees near the origin with the function G introduced above and extends continuously to $u = 0$.

For $0 < k \leq \delta$ and $\beta > 0$, put $a := \omega_{\text{bg}}(k)$ and define

$$\begin{aligned} J_n = J_n(k, \beta) &:= \left\{ t \geq 0 : \frac{t}{\beta}, \frac{t}{\beta} + a \in \omega_{\text{bg}}(I_n), \right. \\ &\quad \left. \left| p_n\left(\frac{t}{\beta}\right) - p_n\left(\frac{t}{\beta} + a\right) \right| \leq k \leq p_n\left(\frac{t}{\beta}\right) + p_n\left(\frac{t}{\beta} + a\right) \right\}. \end{aligned}$$

Let $\lambda_n(t)$ be 1 when both triangle inequalities are strict and $\frac{1}{2}$ when one is an equality. The latter value is the boundary weight for when the zero of the Dirac delta distribution lies exactly on the boudary of the integration interval.

Then

$$\begin{aligned} \gamma_L(\mathbf{k}; \beta, \nu) &= \frac{\hat{v}(0)}{32\pi k \omega_{\text{bg}}(k)} \frac{1}{\beta \nu} \sum_{n=0}^N \int_{J_n} dt \lambda_n(t) G_n\left(\frac{t}{\beta}, \omega_{\text{bg}}(k)\right) \\ &\quad \times f_n\left(\frac{t}{\beta}\right) f_n\left(\frac{t}{\beta} + \omega_{\text{bg}}(k)\right) \frac{(e^{\beta \omega_{\text{bg}}(k)} - 1) e^t}{(e^t - 1)(e^{t + \beta \omega_{\text{bg}}(k)} - 1)}. \end{aligned} \quad (3.91)$$

Both factors f_n are bounded in absolute value on these domains, uniformly in $0 < k \leq \delta$ and $\beta > 0$. Furthermore, there are constants $b_0 > 0$ and $b_n \geq b_0$ for $1 \leq n \leq N$, independent of k and β , such that

$$[0, \beta \nu b_0] \subset J_0 \subset [0, +\infty[, \quad \lambda_0(t) = 1 \quad \text{for } 0 < t \leq \beta \nu b_0,$$

and

$$J_n \subset [\beta \nu b_n, +\infty[, \quad 1 \leq n \leq N.$$

Proof. It is convenient to work with radial coordinates rather than the resonance set in $\mathbb{R}^3$. Set

$$\mathcal{R}_k := \{(p, q) \in [0, +\infty[^2 : |p - q| \leq k \leq p + q, \quad \omega_{\text{bg}}(q) = \omega_{\text{bg}}(p) + \omega_{\text{bg}}(k)\}.$$

The triangle inequalities are precisely the conditions for the existence of a vector $\mathbf{p}$ with $|\mathbf{p}| = p$ and $|\mathbf{p} - \mathbf{k}| = q$. Thus Lemma 3.9 gives

$$[0, K] \subset \{p \geq 0 : \text{there exists } q \geq 0 \text{ such that } (p, q) \in \mathcal{R}_k\}. \quad (3.92)$$

By (3.5b), ω'_{bg} has only finitely many zeros, and there are $R, \epsilon_\infty > 0$ such that

$$\frac{\omega_{\text{bg}}(r) \omega'_{\text{bg}}(r)}{r} = \frac{r^2}{2} + \frac{\nu}{\hat{v}(0)} \left(\hat{v}(r) + \frac{r}{2} \hat{v}'(r) \right) \geq \epsilon_\infty, \quad r \geq R. \quad (3.93)$$

Thus, all stationary points lie in $[0, R]$. Apply Lemma 3.10 to the compact set $[0, R + 1]$ and choose $\delta \leq 1$ accordingly. For every resonant pair (p, q) for which $p \in [0, R + 1]$, it follows that

$$\min_{r \in [\min\{p, q\}, \max\{p, q\}]} |\omega'_{\text{bg}}(r)| > \epsilon \sqrt{\nu}. \quad (3.94)$$

If $p > R + 1$, then $q > R$ and the interval between p and q lies in the increasing tail. Consequently, every resonant pair $(p, q) \in \mathcal{R}_k$ belongs to $I_n \times I_n$ for a single n .

Since $\hat{v}$ is bounded, $\omega_{\text{bg}}(r) \geq C \frac{r^2}{2}$ as $r \rightarrow +\infty$ for some $C > 0$. Together with $\frac{\omega_{\text{bg}}(r)}{r} \rightarrow \sqrt{\nu}$ at zero and $\omega_{\text{bg}}(r) > 0$ for $r > 0$, this implies that

$$\frac{r}{\omega_{\text{bg}}(r)} \leq \frac{C}{\sqrt{\nu}}, \quad r > 0,$$

for some $C > 0$. Therefore, (3.93) and (3.94) yield

$$|f_n(u)| = \frac{p_n(u)}{u |\omega'_{\text{bg}}(p_n(u))|} \leq \max \left\{ \frac{C}{\epsilon \nu}, \frac{1}{\epsilon_\infty} \right\}, \quad p_n(u) \in I_n. \quad (3.95)$$

The same estimate holds at $u + \omega_{\text{bg}}(k)$.

Integrating out the azimuthal angle and changing from the polar angle to q gives the measure

$$\frac{2\pi p q}{k} dp dq.$$

On $I_n \times I_n$, set $u = \omega_{\text{bg}}(p)$ and $w = \omega_{\text{bg}}(q)$. The absolute value of the Jacobian is

$$p q dp dq = u w |f_n(u) f_n(w)| du dw = u w f_n(u) f_n(w) du dw,$$

because the two inverse derivatives have the same sign on a single interval. The delta distribution is $\delta(w - u - a)$, where $a = \omega_{\text{bg}}(k) > 0$. On its support, the thermal factor simplifies to

$$\frac{(e^{\frac{\beta}{2}(a+w)} - e^{\frac{\beta}{2}u})^2}{(e^{\beta a} - 1)(e^{\beta u} - 1)(e^{\beta w} - 1)} = \frac{(e^{\beta a} - 1)e^{\beta u}}{(e^{\beta u} - 1)(e^{\beta(u+a)} - 1)}.$$

Using (3.90), evaluating the w integral, and setting $t = \beta u$ gives (3.91), with the boundary weights $\lambda_n(t)$ specified in the statement. In the coordinates x, y of (3.18)–(3.19), these steps amount to setting $y = -a$ and

$$t = \frac{\beta}{2}(x - a) = \beta\nu \frac{x - \omega_{\text{bg}}(k)}{2\nu},$$

in agreement with the rescaling introduced above.

Finally, choose K as in Lemma 3.9 and decrease δ so that $[0, K + \delta] \subset I_0$. For every $p \in [0, K]$, a resonant q exists and satisfies $q \leq p + k \leq K + \delta$. Thus, $(p, q) \in I_0 \times I_0$ and

$$[0, \beta\omega_{\text{bg}}(K)] \subset J_0.$$

The strict inequalities in the proof of lemma 3.9 also imply $\lambda_0 = 1$ on this interval away from zero. If $N \geq 1$, each I_n , $n \geq 1$, is bounded away from zero. Hence

$$m_n := \inf_{r \in I_n} \omega_{\text{bg}}(r) > 0.$$

Set $b_n = \frac{m_n}{\nu}$ for $n \geq 1$ and choose

$$0 < b_0 \leq \min \left\{ \frac{\omega_{\text{bg}}(K)}{\nu}, b_1, \dots, b_N \right\},$$

omitting the b_n when $N = 0$. These constants give the asserted inclusions and are independent of k and β . $\square$

We can now state and prove the theorem estimating the Landau damping rate.

Theorem 3.12. *Fix a $\nu_0 \in]0, +\infty[$ and suppose the potential satisfies (2.2a)–(3.5b) for any $\nu \in]0, \nu_0]$. Then, in the limit of small temperature $\frac{1}{\beta\nu} \rightarrow 0$ and small momenta $\frac{|\mathbf{k}|}{\sqrt{\nu}} \rightarrow 0$:*

(a) *the Landau damping rate is equal to*

$$\begin{aligned} \gamma_{\text{L}}(\mathbf{k}; \beta, \nu) &= \frac{9\hat{v}(\mathbf{0})\nu^{3/2}}{64\pi} \frac{1}{(\beta\nu)^5} \left[\mathcal{G}_4(\beta\omega_{\text{bg}}(\mathbf{k})) \right. \\ &\quad + 2\beta\sqrt{\nu}|\mathbf{k}|\mathcal{G}_3(\beta\omega_{\text{bg}}(\mathbf{k})) \\ &\quad \left. + (\beta\sqrt{\nu}|\mathbf{k}|)^2\mathcal{G}_2(\beta\omega_{\text{bg}}(\mathbf{k})) \right] \left(1 + O\left(\frac{1}{(\beta\nu)^2} + \frac{|\mathbf{k}|^2}{\nu} \right) \right) \end{aligned} \quad (3.96)$$

$$\text{as } \frac{|\mathbf{k}|}{\sqrt{\nu}}, \frac{1}{\beta\nu} \rightarrow 0 \quad (3.97)$$

where

$$\mathcal{G}_k(\theta) = \int_0^{+\infty} dt \frac{\sinh(\frac{\theta}{2})t^k}{\sinh(\frac{t+\theta}{2})\sinh(\frac{t}{2})} \quad (3.98)$$

is a continuous function of $\theta \in [0, +\infty)$ satisfying

$$\mathcal{G}_k(\theta) = 2\Gamma(k+1)\zeta(k)\theta + O(\theta^2) \quad \text{as } \theta \rightarrow 0, \quad (3.99)$$

$$\mathcal{G}_k(\theta) = 2\Gamma(k+1)\zeta(k+1) + O(e^{-\theta}) \quad \text{as } \theta \rightarrow +\infty. \quad (3.100)$$

(b) Additionally, in the limit $\beta\sqrt{\nu}|\mathbf{k}| \rightarrow 0$, the Landau damping rate can be estimated as

$$\gamma_{\text{L}}(\mathbf{k}; \beta, \nu) = \frac{3\pi^3 \hat{v}(\mathbf{0}) \nu^{3/2}}{40} \frac{|\mathbf{k}|}{\sqrt{\nu}} \frac{1}{(\beta\nu)^4} \left(1 + O\left(\beta\sqrt{\nu}|\mathbf{k}| + \frac{1}{(\beta\nu)^2} \right) \right) \\ \text{as } \frac{|\mathbf{k}|}{\sqrt{\nu}}, \frac{1}{\beta\nu}, \beta\sqrt{\nu}|\mathbf{k}| \rightarrow 0 \quad (3.101)$$

(c) In the opposite limit, $\frac{1}{\beta\sqrt{\nu}|\mathbf{k}|} \rightarrow 0$, the Landau damping rate can be estimated as

$$\gamma_{\text{L}}(\mathbf{k}; \beta, \nu) = \frac{9\zeta(3) \hat{v}(\mathbf{0}) \nu^{3/2}}{16\pi} \frac{1}{(\beta\nu)^3} \frac{|\mathbf{k}|^2}{\nu} \left(1 + O\left(\frac{1}{\beta\sqrt{\nu}|\mathbf{k}|} + \frac{|\mathbf{k}|^2}{\nu} \right) \right) \\ \text{as } \frac{|\mathbf{k}|}{\sqrt{\nu}}, \frac{1}{\beta\nu}, \frac{1}{\beta\sqrt{\nu}|\mathbf{k}|} \rightarrow 0 \quad (3.102)$$

Proof. (a) We define

$$\delta := \frac{\omega_{\text{bg}}(k)}{\nu}, \quad \theta = \beta\omega_{\text{bg}}(k). \quad (3.103)$$

Then, using Proposition 3.11 we can write the Landau damping integral as

$$\sum_{n=0}^N \frac{\hat{v}(0)}{32\pi k \omega_{\text{bg}}(k)} \frac{1}{\beta\nu} \int_0^{+\infty} dt \chi_{J_n}(t) G\left(\frac{\nu t}{\beta\nu}, \nu\delta\right) f_n\left(\frac{\nu t}{\beta\nu}\right) f_n\left(\frac{\nu t + \nu\theta}{\beta\nu}\right) \\ \times \frac{(e^\theta - 1)e^t}{(e^t - 1)(e^{t+\theta} - 1)}, \quad (3.104)$$

where χ_{J_n} is the characteristic function of the set J_n introduced in the same Proposition (3.11). Without loss of generality, we can suppose that $[0, \beta\nu] \subset J_0$. Now, we separate the integral into two regions of integration

$$\int_0^{+\infty} = \int_0^{\beta\nu} + \int_{\beta\nu}^{+\infty}, \quad (3.105)$$

Integral for $t \leq \beta\nu$. Let us write G as in (3.41) and expand f around 0, so to obtain

$$\frac{\hat{v}(0)}{32\pi k \omega_{\text{bg}}(k) \nu^2} \frac{1}{\beta\nu} \int_0^{\beta\nu} dt \left[9\nu^5 \delta^2 \frac{t^2}{(\beta\nu)^2} \left(\delta + \frac{t}{\beta\nu} \right)^2 \right. \\ \left. + \frac{t^2}{(\beta\nu)^2} \delta^2 S\left(\frac{\nu t}{\beta\nu}, \nu\delta\right) \right] \frac{(e^\theta - 1)e^t}{(e^t - 1)(e^{t+\theta} - 1)} \left(1 + O\left(\frac{t^2}{(\beta\nu)^2} \right) \right). \quad (3.106)$$

Let us first focus on the leading term on in Equation (3.106). This can be written as

$$\int_0^{\beta\nu} dt \delta^2 \frac{t^2}{(\beta\nu)^2} \left(\delta + \frac{t}{\beta\nu} \right)^2 \frac{(e^\theta - 1)e^t}{(e^t - 1)(e^{t+\theta} - 1)} \\ = \int_0^{+\infty} dt \delta^2 \frac{t^2}{(\beta\nu)^2} \left(\delta + \frac{t}{\beta\nu} \right)^2 \frac{(e^\theta - 1)e^t}{(e^t - 1)(e^{t+\theta} - 1)} (1 + O(e^{-\frac{\beta\nu}{2}})), \quad (3.107)$$

where we have estimated the remainder as in Lemma 3.8. We know that

$$S\left(\frac{\nu t}{\beta\nu}, \nu\delta\right) = O\left(\left(\frac{t}{\beta\nu} + \delta \right)^4 \right), \quad \text{as } \delta, \frac{t}{\beta\nu} \rightarrow 0, \quad (3.108)$$

thus the remainder term in (3.106) can be estimated directly using Lemma 3.8. This gives a term of order

$$\frac{\hat{v}(0)}{k \omega_{\text{bg}}(k) \nu^2} \frac{1}{\beta\nu} \int_0^{+\infty} dt \nu^5 \delta^2 \frac{t^2}{(\beta\nu)^2} \left(\delta + \frac{t}{\beta\nu} \right)^4 \frac{(e^\theta - 1)e^t}{(e^t - 1)(e^{t+\theta} - 1)}. \quad (3.109)$$

Integral for $t \geq \beta\nu$. This integration region produces a contribution which is exponentially damped as $\frac{1}{\beta\nu} \rightarrow 0$. Indeed, following Lemma 3.3 we can write

$$G\left(\frac{\nu t}{\beta\nu}, \nu\delta\right) = \nu^2 \delta^2 \frac{t^2}{(\beta\nu)^2} K\left(\frac{\nu t}{\beta\nu}, \nu\delta\right) \quad (3.110)$$

where K is continuous and polynomially bounded in its arguments. In addition, the function f is continuous and bounded by Proposition 3.11, Equation (3.95). Thus, following again Lemma 3.8, we have the bound

$$\begin{aligned} & \left| \int_{\beta\nu}^{+\infty} dt \chi_{J_n}(t) G\left(\frac{\nu t}{\beta\nu}, \nu\delta\right) f_n\left(\frac{\nu t}{\beta\nu}\right) f_n\left(\frac{\nu t + \nu\theta}{\beta\nu}\right) \frac{(e^\theta - 1)e^t}{(e^t - 1)(e^{t+\theta} - 1)} \right| \\ & \leq e^{-\frac{\beta\nu}{2}} \delta^2 \nu^2 \int_0^{+\infty} \frac{t^2}{(\beta\nu)^2} P\left(\frac{t}{\beta\nu}\right) \frac{(e^\theta - 1)e^t}{(e^t - 1)(e^{t+\theta} - 1)}, \end{aligned} \quad (3.111)$$

where $t \rightarrow P(t)$ is a suitable polynomial whose coefficients do not depend on $\beta\nu$ nor on δ .

All the integrals in the preceding discussion can be re-arranged as

$$\mathcal{G}_k(\theta) := \int_0^{+\infty} dt \frac{\sinh(\frac{\theta}{2})t^k}{\sinh(\frac{t}{2})\sinh(\frac{t+\theta}{2})}, \quad (3.112)$$

for some integer $k \geq 0$. The integrals defining the special functions $\mathcal{G}_k$ can be computed explicitly in terms of the polylogarithm functions of integer order Li_n via an expansion of the hyperbolic sine

$$\begin{aligned} \mathcal{G}_k(\theta) &= \int_0^{+\infty} dt \frac{\sinh(\frac{\theta}{2})t^k}{\sinh(\frac{t}{2})\sinh(\frac{t+\theta}{2})} \\ &= 4 \sinh(\frac{\theta}{2}) e^{-\frac{\theta}{2}} \int_0^{+\infty} dt \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} e^{-m\theta} e^{-t(n+m+1)} t^k \\ &= 2\Gamma(k+1) \sum_{p=1}^{+\infty} \frac{1}{p^{k+1}} (1 - e^{-p\theta}) \\ &= 2\Gamma(k+1) (\text{Li}_{k+1}(1) - \text{Li}_{k+1}(e^{-\theta})). \end{aligned} \quad (3.113)$$

In particular, Equation (3.113) gives a bounded function of θ for every $k \geq 2$ with asymptotic values

$$\mathcal{G}_k(\theta) = 2\Gamma(k+1)\zeta(k)\theta + O(\theta^2) \quad \text{as } \theta \rightarrow 0, \quad (3.114)$$

$$\mathcal{G}_k(\theta) = 2\Gamma(k+1)\zeta(k+1) + O(e^{-\theta}) \quad \text{as } \theta \rightarrow +\infty. \quad (3.115)$$

Then, the Landau damping can be written for $\frac{|\mathbf{k}|}{\sqrt{\nu}}, \frac{1}{\beta\nu} \rightarrow 0$ as

$$\begin{aligned} \gamma_L(\mathbf{k}; \beta, \nu) &= \frac{9\hat{v}(\mathbf{0})\nu^{3/2}}{64\pi} \frac{1}{(\beta\nu)^5} \left[\mathcal{G}_4(\beta\omega_{\text{bg}}(\mathbf{k})) \right. \\ & \quad + 2\beta\sqrt{\nu}|\mathbf{k}|\mathcal{G}_3(\beta\omega_{\text{bg}}(\mathbf{k})) \\ & \quad \left. + (\beta\sqrt{\nu}|\mathbf{k}|)^2 \mathcal{G}_2(\beta\omega_{\text{bg}}(\mathbf{k})) \right] \left(1 + O\left(\frac{1}{(\beta\nu)^2} + \frac{|\mathbf{k}|^2}{\nu}\right) \right) \end{aligned} \quad (3.116)$$

where we have expanded $\delta = \frac{\omega_{\text{bg}}(\mathbf{k})}{\nu} = \frac{|\mathbf{k}|}{\sqrt{\nu}} + O\left(\frac{|\mathbf{k}|^3}{\nu^{3/2}}\right)$. This concludes the proof of (a).

Points (b) and (c) are immediately verified by exploiting the asymptotic properties of $\mathcal{G}_k$ in Equations (3.114), (3.115). $\square$

3.6. High temperature results. The description of the high temperature limit for the integrals (1.7) and (1.8) requires a careful physical analysis of the scales at play. For the Beliaev damping, the only dependence on β in Equation (3.59) is through the product $\beta\omega_{\text{bg}}(\mathbf{k})$. Thus, while the limit $\beta\omega_{\text{bg}}(\mathbf{k}) \rightarrow +\infty$ necessarily corresponds to a small temperature limit, the opposite limit $\beta\omega_{\text{bg}}(\mathbf{k}) \rightarrow 0$ can be achieved for $\frac{|\mathbf{k}|}{\sqrt{\nu}} \rightarrow 0$, compatibly with the hypotheses $T < T_c$; where T_c is the critical temperature relative to Bose-Einstein phase transition. Similarly, for the Landau damping, the limit $\beta\omega_{\text{bg}}(\mathbf{k}) \rightarrow 0$ could in principle be tested.

Now, we analyze the limit $\beta\nu \rightarrow 0$. In this regime, the parameters have to be chosen so that the assumption $T < T_c$ can still be verified. If we substitute for the density of particles $\frac{n_0(\beta)}{L^3}$ its value for a free Bose gas, we can estimate the critical temperature as

$$T_c \sim \left(\frac{n_0}{L^d} \right)^{\frac{2}{3}}.$$

Then, since $\nu = \hat{v}(\mathbf{0})n_0$, the requirement for $\beta\nu$ to be small is equivalent to

$$\frac{\beta^{\frac{3}{2}} \hat{v}(\mathbf{0}) \nu^{\frac{1}{2}}}{\beta_c^{\frac{3}{2}} (\beta\nu)^{\frac{1}{2}}} \ll 1.$$

This condition could be valid in the limit of a very *dilute* potential.

However, numerical analysis, [53, Fig. 1] and [32, Fig. 1], show that the convergence towards this asymptotic regime is rather slow, requiring very large values of the temperature. Moreover, for high temperatures the Landau damping rate depends on the high momenta region of $\hat{v}$, so that its computation becomes highly potential-dependent.

To compare our results with the pre-existing literature we compute this rate assuming the specific form for the potential

$$\hat{v}(\mathbf{k}) = \begin{cases} \hat{v}(\mathbf{0}) > 0 & |\mathbf{k}| < \Lambda \\ 0 & |\mathbf{k}| \geq 2\Lambda \end{cases}, \quad (3.117)$$

and $\hat{v}(\mathbf{k})$ smoothly interpolating between $\hat{v}(\mathbf{0})$ and 0 for $\Lambda < |\mathbf{k}| < 2\Lambda$. In the computation of the rate, we will remove the cut-off by taking the limit $\Lambda \rightarrow +\infty$. The final expression for the Landau damping rate coincides with the one estimated in the literature [35, 53, 56].

Proposition 3.13. *Suppose the potential is of the form (3.117). Then, after having performed the limit $\Lambda \rightarrow +\infty$, we find*

$$\gamma_{\text{L}}(\mathbf{k}; \beta, \nu) = \frac{3\hat{v}(\mathbf{0})\nu^{3/2}}{32} \frac{1}{\beta\nu} \frac{|\mathbf{k}|}{\sqrt{\nu}} \left(1 + O\left(\frac{|\mathbf{k}|^2}{\nu} + \beta\nu \right) \right) \quad \text{as } \frac{|\mathbf{k}|}{\sqrt{\nu}}, \beta\nu \rightarrow 0. \quad (3.118)$$

Proof. In the high temperature limit, the contribution of momenta outside the phonon regime must be retained. We work in the flat Fourier-transform limit $\hat{v}(\mathbf{k}) = \hat{v}(\mathbf{0})$, after removing the cutoff $\Lambda \rightarrow +\infty$ at fixed $k, \beta, \nu > 0$. This can be justified by a simple dominated convergence argument, using the exponential decay of the thermal factor.

In this limit, the Bogoliubov dispersion relation becomes

$$\omega_{\text{bg}}(k) = \sqrt{\frac{k^4}{4} + \nu k^2}. \quad (3.119)$$

It is strictly increasing for $k \geq 0$, with inverse

$$k(\omega) = \sqrt{-2\nu + 2\sqrt{\nu^2 + \omega^2}}. \quad (3.120)$$

We may therefore introduce the variables x and y as in (3.18) and (3.19). Setting

$$z := \frac{x - \omega_{\text{bg}}(k)}{2\nu}, \quad \delta := \frac{\omega_{\text{bg}}(k)}{\nu}, \quad \theta := \beta\omega_{\text{bg}}(k) = \beta\nu\delta, \quad (3.121)$$

we obtain

$$\begin{aligned} \gamma_{\text{L}}(\mathbf{k}; \beta, \nu) &= \frac{\hat{v}(\mathbf{0})}{32\pi k \omega_{\text{bg}}(k)} \int_0^{+\infty} dz G(\nu z, \nu\delta) f(\nu z) f(\nu z + \nu\delta) \\ &\quad \times \frac{(e^\theta - 1) e^{\beta\nu z}}{(e^{\beta\nu z} - 1)(e^{\beta\nu z + \theta} - 1)}. \end{aligned} \quad (3.122)$$

The flat Fourier transform also gives

$$f(\nu z) = \frac{1}{\nu\sqrt{1+z^2}}, \quad f(\nu z + \nu\delta) = \frac{1}{\nu\sqrt{1+(z+\delta)^2}}, \quad (3.123)$$

and

$$\begin{aligned} G(\nu z, \nu\delta) &= 4\nu^2 \left[c(\nu z + \nu\delta) c(\nu z) c(\nu\delta) - s(\nu z + \nu\delta) s(\nu z) s(\nu\delta) \right. \\ &\quad + s(\nu z + \nu\delta) s(\nu\delta) c(\nu z) - c(\nu z + \nu\delta) c(\nu\delta) s(\nu z) \\ &\quad \left. + s(\nu z + \nu\delta) s(\nu z) c(\nu\delta) - c(\nu z + \nu\delta) c(\nu z) s(\nu\delta) \right]^2. \end{aligned} \quad (3.124)$$

To obtain a low-momentum expansion uniform in $\beta\nu$, we first center the integral by replacing z with $z - \frac{\delta}{2}$, so that the new integration interval is $[\frac{\delta}{2}, +\infty[$. A direct expansion as $\delta \rightarrow 0$ of the centered expression gives, for $z \geq \delta$,

$$\frac{G(\nu(z - \frac{\delta}{2}), \nu\delta) f(\nu(z - \frac{\delta}{2})) f(\nu(z + \frac{\delta}{2}))}{8\nu^3 \delta^2} \quad (3.125)$$

$$= \frac{1}{2} \left(2 - \frac{1}{\sqrt{1+z^2}} - \frac{1}{1+z^2} \right)^2 + O(\delta^2 \min\{z^2, 1\}). \quad (3.126)$$

The remainder is uniform in $z \geq \delta$. The absence of a linear term in δ follows from the fact that the centered expression is even in δ .

The normalized thermal factor satisfies

$$\begin{aligned} \frac{\beta\nu}{\delta} \left(\frac{1}{e^{\beta\nu(z - \frac{\delta}{2})} - 1} - \frac{1}{e^{\beta\nu(z + \frac{\delta}{2})} - 1} \right) &= \frac{1}{\delta} \int_{z - \frac{\delta}{2}}^{z + \frac{\delta}{2}} dx \frac{(\beta\nu)^2}{4 \sinh^2 \frac{\beta\nu x}{2}} \\ &= \frac{(\beta\nu)^2}{4 \sinh^2 \frac{\beta\nu z}{2}} + O\left(\frac{\delta^2}{z^4}\right), \quad z \geq \delta, \end{aligned} \quad (3.127)$$

where we used that the linear term in δ vanishes due to the symmetry under $\delta \rightarrow -\delta$ and the uniform estimates

$$0 < \frac{(\beta\nu)^2}{4 \sinh^2 \frac{\beta\nu z}{2}} \leq \frac{1}{z^2}, \quad \frac{d^2}{dz^2} \left(\frac{(\beta\nu)^2}{4 \sinh^2 \frac{\beta\nu z}{2}} \right) = O(\delta^0 z^{-4}) \text{ as } \delta \rightarrow 0, \quad (3.128)$$

with a constant independent of $\beta\nu$.

Thus, on $[\delta, +\infty)$, the preceding estimates give an error, as $\delta \rightarrow 0$, bounded by

$$O\left(\delta^2 \int_{\delta}^{+\infty} dz \left[\frac{\min\{z^2, 1\}}{z^2} + \frac{\delta^2 \min\{z^2, 1\}}{z^4} \right]\right) = O(\delta^2), \quad (3.129)$$

uniformly in $\beta\nu > 0$.

For $\frac{\delta}{2} \leq z \leq \delta$, the normalized expression (3.125) involving G and f is $O((z^2 - \frac{\delta^2}{4})^2)$, while the thermal factor is bounded above by $(z^2 - \frac{\delta^2}{4})^{-1}$. Therefore, the total contribution to the integral is $O(\delta^3)$.

we conclude that

$$\gamma_L(\mathbf{k}; \beta, \nu) = \frac{\hat{v}(\mathbf{0})\nu^{3/2}}{4\pi} \frac{1}{\beta\nu} \frac{k}{\sqrt{\nu}} \left[\int_0^{+\infty} \frac{dz}{2z^2} \left(2 - \frac{1}{\sqrt{1+z^2}} - \frac{1}{1+z^2} \right)^2 \right. \\ \left. \times \left(\frac{\beta\nu z}{2 \sinh \frac{\beta\nu z}{2}} \right)^2 + O(\delta^2) \right],$$

with a remainder uniform in $\beta\nu > 0$.

We now estimate the high-temperature error. Since

$$0 \leq \frac{1}{2} \left(2 - \frac{1}{\sqrt{1+z^2}} - \frac{1}{1+z^2} \right)^2 \leq 2,$$

we have

$$0 \leq \int_0^{+\infty} \frac{dz}{2z^2} \left(2 - \frac{1}{\sqrt{1+z^2}} - \frac{1}{1+z^2} \right)^2 \left[1 - \left(\frac{\beta\nu z}{2 \sinh(\beta\nu z/2)} \right)^2 \right] \\ \leq 2 \int_0^{+\infty} dz \left[\frac{1}{z^2} - \frac{(\beta\nu)^2}{4 \sinh^2(\beta\nu z/2)} \right] = \beta\nu.$$

Thus the temperature error in the normalized integral is $O(\beta\nu)$.

Finally, the limiting integral is

$$\int_0^{+\infty} \frac{dz}{2z^2} \left(2 - \frac{1}{\sqrt{1+z^2}} - \frac{1}{1+z^2} \right)^2 \\ = \left[\frac{3}{4} \arctan z + \frac{1}{z} \left(\frac{1}{\sqrt{1+z^2}} - 1 \right) - \frac{z}{4(1+z^2)} \right]_0^{+\infty} = \frac{3\pi}{8}. \quad (3.130)$$

Combining these estimates yields

$$\gamma_L(\mathbf{k}; \beta, \nu) = \frac{3\hat{v}(\mathbf{0})\nu^{3/2}}{32} \frac{1}{\beta\nu} \frac{|\mathbf{k}|}{\sqrt{\nu}} \left(1 + O \left(\frac{|\mathbf{k}|^2}{\nu} + \beta\nu \right) \right), \\ \frac{|\mathbf{k}|}{\sqrt{\nu}}, \beta\nu \rightarrow 0, \quad (3.131)$$

uniformly, without any restriction on the relative sizes of $\frac{|\mathbf{k}|}{\sqrt{\nu}}$ and $\beta\nu$. $\square$

Remark 3.14. In addition to the previous consideration regarding the well-foundedness of the hypotheses $T < T_c$, we observe how Equation (3.118) is divergent to the leading order as $\beta\nu \rightarrow 0$, which can be seen as a manifestation of the non validity of the high temperature limit.

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REFERENCES

- [1] Abo-Shaeer, J. R., Ketterle, W., Raman, C., Vogels, J. M., Xu, K., *Experimental Observation of the Bogoliubov Transformation for a Bose–Einstein Condensed Gas*. Phys. Rev. Lett. **88**, 6, 060402 (2002).
- [2] Araki, H., Woods, E. J., *Representations of the Canonical Commutation Relations Describing a Nonrelativistic Infinite Free Bose Gas*. J. Math. Phys. **4**, 637 (1963).
- [3] Arnowitt, R., Girardeau, M., *Theory of Many-Boson Systems: Pair Theory*. Phys. Rev. **113**, 3, 755 (1958).
- [4] Beauvois, K., Dawidowski, J., Godfrin, H., Krotscheck, E., Fåk, B., Ollivier, J., Sultan, A., *Dispersion Relation of Landau Elementary Excitations and Thermodynamic Properties of Superfluid ^4He* . Phys. Rev. B **103**, 10, 104516 (2021).
- [5] Beliaev, S. T., *Applications of the Methods of Quantum Field Theory to a System of Bosons*. Sov. Phys. JETP **34**, 290 (1958) (translation of Zh. Eksp. Teor. Fiz. **34**, 7, 417 (1958)).
- [6] Beliaev, S. T., *Energy Spectrum of a Non-Ideal Bose Gas*. Sov. Phys. JETP **34**, 7, 299 (1958) (translation of Zh. Eksp. Teor. Fiz. **34**, 433 (1958)).
- [7] Boccato, C., Brennecke, C., Cenatiempo, S., Schlein, B., *Bogoliubov Theory in the Gross–Pitaevskii Limit*. Acta Math. **222**, 2, 219 (2019).
- [8] Boccato, C., Brennecke, C., Cenatiempo, S., Schlein, B., *The Excitation Spectrum of Bose Gases Interacting Through Singular Potentials*. J. Eur. Math. Soc. **22**, 7, 2331 (2020).
- [9] Bogoliubov, N. N., *On the Theory of Superfluidity*. J. Phys. (USSR) **11**, 23–32 (1947).
- [10] Bossmann L., Petrat S., Seiringer R., *Asymptotic expansion of the low-energy excitation spectrum for weakly interacting bosons*. Forum of Mathematics, Sigma **9**, e28 (2021)
- [11] Brattelli, O., Robinson D. W.: *Operator Algebras and Quantum Statistical Mechanics, Volume 1*, Springer-Verlag, Berlin, second edition 1987.
- [12] Brattelli, O., Robinson D. W.: *Operator Algebras and Quantum Statistical Mechanics, Volume 2*, Springer-Verlag, Berlin, second edition 1996.
- [13] Brennecke C., Caporaletti M, Schlein B., *Excitation spectrum of Bose gases beyond the Gross–Pitaevskii regime*. Rev. Math. Phys. **34** (9), 2250027 (2022)
- [14] Brennecke C., Schlein B., Schraven S., *Bogoliubov Theory for Trapped Bosons in the Gross–Pitaevskii Regime*. Ann. Henri Poincaré, **23**, 1583 (2022)
- [15] Buchholz, D., *The Resolvent Algebra of Nonrelativistic Bose Fields: Sectors, Morphisms, Fields and Dynamics*. Commun. Math. Phys. **375**, 1159 (2020).
- [16] Buchholz, D., Yngvason, J., *Many-Body Physics and Resolvent Algebras*. J. Math. Phys. **66**, 051902 (2025).
- [17] Bryan, M., Sokol, P. E., *Maxon and Roton Measurements in Nanoconfined ^4He* . Phys. Rev. B **97**, 18, 184511 (2018).
- [18] Caraci, C., Olgiaiti, A., Saint-Aubin, D., Schlein, B., *Third Order Corrections to the Ground State Energy of a Bose Gas in the Gross–Pitaevskii Regime*. Commun. Math. Phys. **406**, 153 (2025).
- [19] Cornean, H. D., Dereziński, J., Ziń, P., *On the Infimum of the Energy–Momentum Spectrum of a Homogeneous Bose Gas*. J. Math. Phys. **50** (2009).
- [20] Cornell, E. A., Ensher, J. R., Jin, D. S., Matthews, M. R., Wieman, C. E., *Temperature-Dependent Damping and Frequency Shifts in Collective Excitations of a Dilute Bose–Einstein Condensate*. Phys. Rev. Lett. **78**, 5, 764 (1997).
- [21] Davidson, N., Katz, N., Ozeri, R., Steinhauer, J., *Beliaev Damping of Quasiparticles in a Bose–Einstein Condensate*. Phys. Rev. Lett. **89**, 22, 220401 (2002).
- [22] Dereziński, J., Gérard, C., *Mathematics of Quantization and Quantum Fields*. Cambridge Monographs in Mathematical Physics, Cambridge Univ. Press (2013).
- [23] Dereziński, J., Jakšić, V., Pillet, C.-A., *Perturbation Theory of W^* -Dynamics, Liouvilleans and KMS States*. Rev. Math. Phys. **15**, 447–489 (2003).
- [24] Dereziński, J., Napiórkowski, M., *Excitation Spectrum of Interacting Bosons in the Mean-Field Infinite-Volume Limit*. Ann. Henri Poincaré **15**, 5, 2409 (2014).
- [25] Dereziński, J., Jakšić, V., *Return to equilibrium for Pauli-Fierz systems*, Ann. H. Poincaré **4** (2003) 739–793
- [26] Dereziński, J., Li, B., Napiórkowski, M., *Beliaev Damping in Bose Gas*. J. Stat. Phys. **191**, 110 (2024).
- [27] Dereziński, J., Pettinari, L., *Phonon Dispersion Relation in Weakly Coupled Bose Gas Beyond Bogoliubov Approximation*. In preparation
- [28] Foot, C. J., Hechenblaikner, G., Hodby, E., Maragò, O. M., *Experimental Observation of Beliaev Coupling in a Bose–Einstein Condensate*. Phys. Rev. Lett. **86**, 2196 (2001).
- [29] Galanda S., Pinamonti N., *Equilibrium states for non relativistic Bose gases with condensation*. ArXiv:2509.25101 (2025)

- [30] Galanda S., Pinamonti N., *Equilibrium states for non relativistic Bose gases and the Gross-Pitaevskii limit*. ArXiv:2606.25775 (2026)
- [31] Gavoret, J., Nozières, P., *Structure of the Perturbation Expansion for the Bose Liquid at Zero Temperature*. Ann. Phys. **28**, 3, 349 (1964).
- [32] Giorgini, S., *Damping in Dilute Bose Gases: A Mean-Field Approach*. Phys. Rev. A **57**, 2949 (1998).
- [33] Grech P., Seiringer R., *The excitation spectrum for weakly interacting bosons in a trap*. Comm. Math. Phys. **322**, 559–591 (2013)
- [34] Griffin, A., *Excitations in a Bose-Condensed Liquid*. Cambridge Studies in Low Temperature Physics (2004).
- [35] Griffin, A., Hua, S., *Finite-Temperature Excitations in a Dilute Bose-Condensed Gas*. Phys. Rep. **304**, 1–87 (1998).
- [36] Fetter, A. L., Walecka, J. D., *Quantum Theory of Many-Particle Systems*. Dover Publications (2003).
- [37] Hohenberg, P. C., Martin, P. C., *Microscopic Theory of Superfluid Helium*. Ann. Phys. **34**, 2, 291 (1965).
- [38] Hugenholtz, N. M., Pines, D., *Ground-State Energy and Excitation Spectrum of a System of Interacting Bosons*. Phys. Rev. **116**, 489 (1959).
- [39] Jakšić, V., Pillet, C.-A.: On a model for quantum friction II: Fermi’s golden rule and dynamics at positive temperature, Comm. Math. Phys. **176**, 619 (1996).
- [40] Kurkjian, H., Ristivojevic, Z., *Damping of Elementary Excitations in One-Dimensional Dipolar Bose Gases*. Phys. Rev. Research **2**, 033337 (2020).
- [41] Lee, T. D., Yang, C. N., *Low-Temperature Behavior of a Dilute Bose System of Hard Spheres. I. Equilibrium Properties*. Phys. Rev. **112**, 1419 (1958).
- [42] Lewin, L., *Polylogarithms and Associated Functions*. North-Holland (1981).
- [43] Lewin, M., Nam, P. T., Serfaty, S., Solovej, J. P., *Bogoliubov Spectrum of Interacting Bose Gases*. Commun. Pure Appl. Math. **68**, 413 (2015).
- [44] Lieb, E. H., Seiringer, R., Yngvason, J., *Justification of c-Number Substitutions in Bosonic Hamiltonians*. Phys. Rev. Lett. **94**, 080401 (2005).
- [45] Liu, W. V., *Theoretical Study of the Damping of Collective Excitations in a Bose–Einstein Condensate*. Phys. Rev. Lett. **79**, 21, 4056 (1997).
- [46] Mohling, F., Morita, M., *Temperature Dependence of the Low-Momentum Excitations in a Bose Gas of Hard Spheres*. Phys. Rev. **120**, 681 (1960).
- [47] Mohling, F., Sirlin, A., *Low-Lying Excitations in a Bose Gas of Hard Spheres*. Phys. Rev. **118**, 370 (1959).
- [48] Nam, P. T., Napiórkowski, M., *Norm Approximation for Many-Body Quantum Dynamics and Bogoliubov Theory*. In *Advances in Quantum Mechanics*, Springer (2017).
- [49] Nam, P. T., Napiórkowski, M., *Two-Term Expansion of the Ground-State One-Body Density Matrix of a Mean-Field Bose Gas*. Calc. Var. PDE **60**, 3 (2021).
- [50] Nam, P. T., Seiringer, R., *Collective Excitations of Bose Gases in the Mean-Field Regime*. Arch. Ration. Mech. Anal. **215**, 2, 381 (2015).
- [51] Nam, P. T., Triay, M., *Bogoliubov Excitation Spectrum of Trapped Bose Gases in the Gross–Pitaevskii Regime*. J. Math. Pures Appl. **176**, 18–101 (2023).
- [52] Napiórkowski, M., *Dynamics of Interacting Bosons: A Compact Review*. In *Density Functionals for Many-Particle Systems*. World Scientific (2023).
- [53] Pitaevskii, L. P., Stringari, S., *Landau Damping in Dilute Bose Gases*. Phys. Lett. A **235**, 398–402 (1997).
- [54] Reed, M., Simon, B., *Methods of Modern Mathematical Physics IV: Analysis of Operators*. Academic Press (1978).
- [55] Seiringer, R., *The Excitation Spectrum for Weakly Interacting Bosons*. Commun. Math. Phys. **306**, 565 (2011).
- [56] Szépfalussy, P., Kondor, I., *On the Dynamics of Continuous Phase Transitions*. Ann. Phys. **82**, 1, 1–53 (1974).
- [57] Tai, T. W., *Ground State of a Bose System of Hard Spheres*. Phys. Rev. **115**, 6, 1390 (1959).
- [58] Takano, F., *Excitation Spectrum in Many-Boson Systems*. Phys. Rev. **123**, 2, 701 (1961).

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