9. Active linear optics

instead of $\vec{a}' = B\vec{a}$ ($n \times n$ matrix) (passive linear optics, B - unitary)

We consider more general transformations

$$\begin{pmatrix} \vec{a}' \\ \vec{a}'^{+} \end{pmatrix} = S \begin{pmatrix} \vec{a} \\ \vec{a}^{+} \end{pmatrix}$$

n - number of modes

$2n \times 2n$ matrix

Preservation of commutation relations:

$$\left[\begin{pmatrix} \vec{a}' \\ \vec{a}'^{+} \end{pmatrix}, (\vec{a}'^{+}, \vec{a}') \right] = \begin{pmatrix} [\vec{a}', \vec{a}'^{+}], & [\vec{a}', \vec{a}'] \\ [\vec{a}'^{+}, \vec{a}'^{+}], & [\vec{a}'^{+}, \vec{a}'] \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{G}$$

$$\left[S \begin{pmatrix} \vec{a} \\ \vec{a}^{+} \end{pmatrix}, (\vec{a}^{+}, \vec{a}) S^{+} \right] = S \underbrace{\left[\begin{pmatrix} \vec{a} \\ \vec{a}^{+} \end{pmatrix}, (\vec{a}^{+}, \vec{a}) \right]}_{G} S^{+}$$

$$\boxed{S \cdot G S^{+} = G, \quad G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

S - quasiunitary.

{ in passive case $S = \begin{pmatrix} B & 0 \\ 0 & B^{+} \end{pmatrix}$

As before we are looking for $\hat{U}_S$:

$$\begin{pmatrix} \vec{a}'^{1} \\ \vec{a}'^{1+} \end{pmatrix} = S \begin{pmatrix} \vec{a} \\ \vec{a}^{+} \end{pmatrix} = \hat{U}_S^{+} \begin{pmatrix} \vec{a} \\ \vec{a}^{+} \end{pmatrix} \hat{U}_S$$

Fact

$$\hat{U}_S = e^{i\hat{H}_S}, \qquad \hat{H}_S = \frac{1}{2}\left(\hat{a}^\dagger, \hat{a}\right)\mathcal{H}\begin{pmatrix}\hat{a}\\ \hat{a}^\dagger\end{pmatrix}$$

$$\mathcal{H} = -iG\cdot \ln S \qquad \{\ S = e^{iG\mathcal{H}}$$

$\mathcal{H}$ hermitian

{ let us show that $\mathcal{H}$ is Hermitian.

$$e^{\ln S} = \sum_{k=0}^{\infty}\frac{(\ln S)^k}{k!} = S$$

$$\mathcal{H}^\dagger = i(\ln S)^\dagger \underbrace{G^\dagger}_{G} =$$

{

$$SGS^\dagger = G \Rightarrow S^{-1} = GS^\dagger G$$

$$G(\ln S)^\dagger G = \ln(GS^\dagger G) \Longrightarrow \ln(S^{-1}) = -\ln S$$

$G^2 = \mathbb{1}$

$$G(\ln S)^\dagger G = -\ln S \Rightarrow (\ln S)^\dagger G = -G\ln S$$

$$\mathcal{H}^\dagger = i(\ln S)^\dagger G = -iG\ln S = \mathcal{H}.$$

Examples single-mode squeezing

$$\hat{U}_S = e^{\frac{1}{2}(\zeta^*\hat{a}^2 - \zeta\hat{a}^{\dagger 2})} \qquad \hat{H}_S = \frac{1}{2i}\left(\zeta^*\hat{a}^2 - \zeta\hat{a}^{\dagger 2}\right)$$

$$\mathcal{H} = \frac{1}{i}\begin{pmatrix}0 & -\zeta\\ \zeta^* & 0\end{pmatrix} \qquad \zeta = re^{i\theta}$$

$$S = e^{\underbrace{i\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{G}\cdot\underbrace{\frac{1}{i}\begin{pmatrix}0&-\zeta\\ \zeta^*&0\end{pmatrix}}_{\mathcal{H}}} = e^{\begin{pmatrix}0&-\zeta\\-\zeta^*&0\end{pmatrix}} = \mathbb{1}\cdot\cosh r - \begin{pmatrix}0 & \frac{\zeta}{r}\\ \frac{\zeta^*}{r} & 0\end{pmatrix}\sinh r$$

{

$$\begin{pmatrix}0&-\zeta\\-\zeta^*&0\end{pmatrix} = \begin{pmatrix}|\zeta|^2&0\\0&|\zeta|^2\end{pmatrix} = |\zeta|^2\mathbb{1}$$

$$= \begin{pmatrix}\cosh r & -\sinh r\cdot e^{i\theta}\\ -\sinh r\, e^{-i\theta} & \cosh r\end{pmatrix}$$

– in agreement what we had before.

$$\begin{pmatrix} \hat{a}' \\ \hat{a}'^{\dagger} \end{pmatrix} = \begin{pmatrix} \cosh r & -\sinh r\, e^{i\theta} \\ -\sinh r\, e^{-i\theta} & \cosh r \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^{\dagger} \end{pmatrix}$$

physical intuition - nonlinear optics.

simple Hamiltonian appearing in non-linear optics:

$$H = \hbar\omega\, a^{\dagger}a + \hbar\nu\, b^{\dagger}b + i\hbar\chi^{(2)}\left(\hat{a}^2\hat{b}^{\dagger} - \hat{a}^{\dagger 2}\hat{b}\right)$$

if mode $\hat{b}$ is classical (very strong beam)

then $\hat{b} \to \beta$, interaction part $\sim i\hbar\chi^{(2)}\left(\hat{a}^2\beta^{*} - \hat{a}^{\dagger 2}\beta\right)$

Two-mode squeezing

$$U_S = e^{r\left(a_1^{\dagger}a_2^{\dagger} - a_1 a_2\right)} \qquad \hat{H} = \frac{r}{i}\left(a_1^{\dagger}a_2^{\dagger} - a_1 a_2\right)$$

$$\mathcal{H} = \frac{r}{i}\begin{pmatrix} 0 & 1 & & 1 \\ & 0 & -1 & \\ & -1 & 0 & \\ -1 & & & 0 \end{pmatrix} \qquad S = e^{r\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} =$$

$$= e^{r\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = \cosh r\, \mathbb{1} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sinh r$$

$$\begin{pmatrix} a_1' \\ a_2' \\ a_1'^{\dagger} \\ a_2'^{\dagger} \end{pmatrix} = \begin{pmatrix} \cosh r & & & \sinh r \\ & \cosh r & \sinh r & \\ & \sinh r & \cosh r & \\ \sinh r & & & \cosh r \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_1^{\dagger} \\ a_2^{\dagger} \end{pmatrix}$$

If we act on the vacuum state

$$|r\rangle_{12} = U_S\,|0\rangle|0\rangle = \ldots = \frac{1}{\cosh r}\sum_{n=0}^{\infty}(\tanh r)^n\,|n,n\rangle$$

↑ two mode squeezed vacuum state

↑ very entangled state (for large r)

If we look at the reduced density matrix of a single mode:

$$\rho = Tr_2\left(|r\rangle_{12}\langle r|\right) = \frac{1}{(\cosh r)^2}\sum_{n,n'}(\tanh r)^{n+n'}|n\rangle_1\langle n'|\otimes|n\rangle_2\langle n'|$$

$$Tr_2$$

$$= \frac{1}{\cosh^2 r}\sum_n (\tanh r)^{2n}|n\rangle\langle n| = \sum_n \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}|n\rangle\langle n|$$

$$\bar{n} = \sinh^2 r \qquad \{\cosh^2 r - \sinh^2 r = 1$$

:

Linear optical amplifier:

We would like to have a device that

thus $|\alpha\rangle \longrightarrow |\sqrt{g}\,\alpha\rangle$, $g > 1$

this we would do if we had a

transformation that makes:

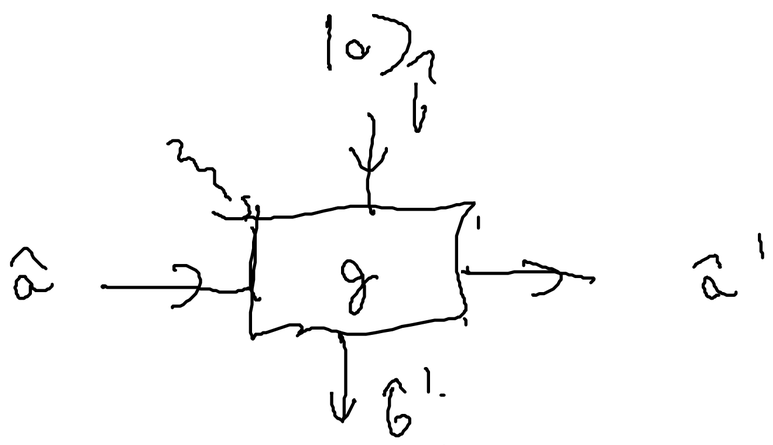
$$\hat{a}' = \sqrt{g}\,\hat{a}$$ - impossible (does not preserve commutation relations)

$\cosh r$ $\sinh r$

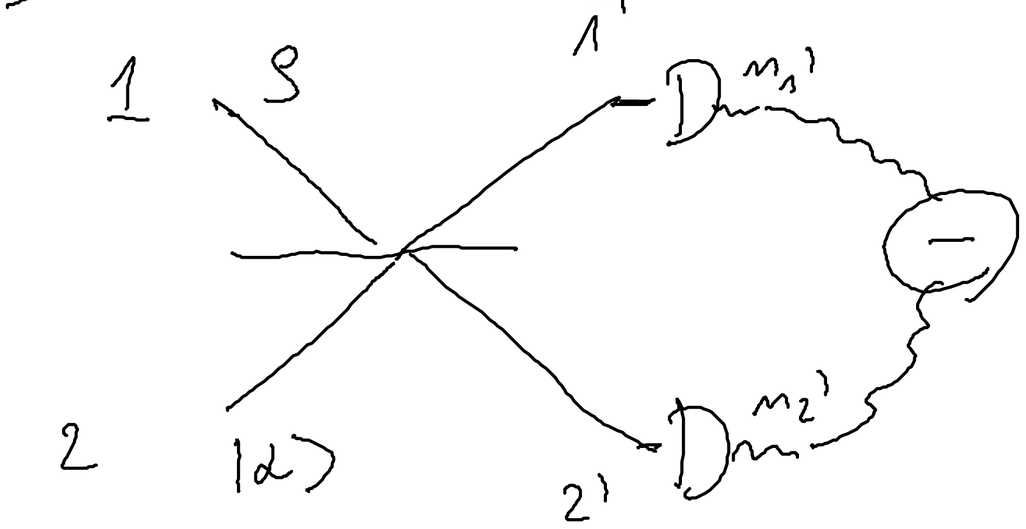
$$\hat{a}' = \sqrt{g}\,\hat{a} + \sqrt{g-1}\,\hat{b}^\dagger$$

↑ additional mode

(resulting in additional noise)



Homodyne detection – measure a quadrature in mode 1



$$\begin{bmatrix} a_1' \\ a_2' \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

We look at an observable:

$$\hat{A} = \hat{n}_1' - \hat{n}_2' = \hat{a}_1'^{\dagger}\hat{a}_1' - \hat{a}_2'^{\dagger}\hat{a}_2' =$$

$$= \frac{1}{2}\left[(\hat{a}_1^{\dagger} + \hat{a}_2^{\dagger})(\hat{a}_1 + \hat{a}_2) - (-\hat{a}_1^{\dagger} + \hat{a}_2^{\dagger})(-\hat{a}_1 + \hat{a}_2) \right] =$$

$$= \hat{a}_1^{\dagger}\hat{a}_2 + \hat{a}_2^{\dagger}\hat{a}_1 \quad \left\{ \approx \hat{a}_1^{\dagger}\alpha + \alpha^* \hat{a}_1 \right.$$
in the limit of strong α

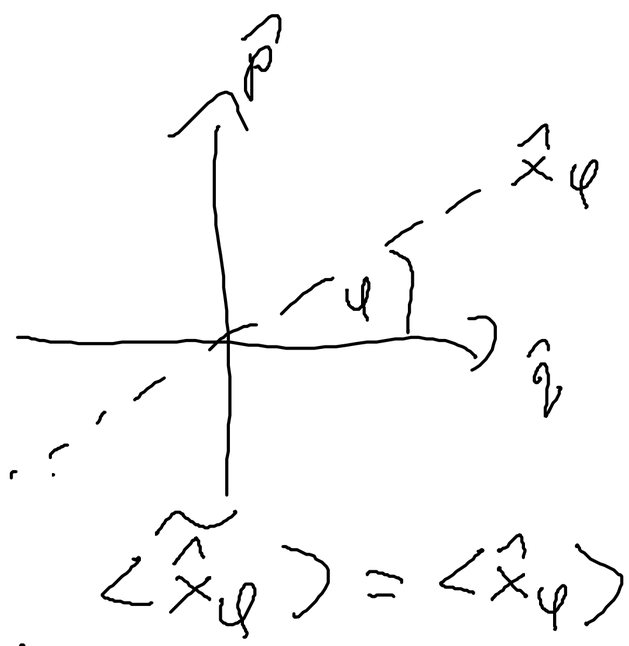
Mean value:

$$\langle \hat{A} \rangle = \mathrm{Tr}_{12}\left(\hat{A} \cdot S \otimes |\alpha\rangle\langle\alpha| \right) = \mathrm{Tr}_1\left[S \left(\hat{a}_1^{\dagger}\alpha + \alpha^* \hat{a}_1 \right) \right]$$

$$\alpha = |\alpha| e^{i\theta}$$

$$= \sqrt{2}|\alpha| \, \mathrm{Tr}\left(S \underbrace{\frac{\hat{a}_1^{\dagger} e^{i\varphi} + \hat{a}_1 e^{-i\varphi}}{\sqrt{2}}}_{\hat{x}_\varphi} \right)$$

$$\langle \hat{A} \rangle = \sqrt{2}|\alpha| \, \langle \hat{x}_\varphi \rangle_S \,, \qquad \tilde{\hat{x}}_\varphi = \frac{\hat{A}}{\sqrt{2}|\alpha|} \,, \qquad \langle \tilde{\hat{x}}_\varphi \rangle = \langle \hat{x}_\varphi \rangle$$



Second moment:

$$\langle \hat{A}^2 \rangle = \langle \left(a_1^{\dagger} a_2 + a_2^{\dagger} a_1 \right)\left(a_1^{\dagger} a_2 + a_2^{\dagger} a_1 \right) \rangle =$$

$$= \langle a_1^{\dagger 2} a_2^2 + a_1^{\dagger} a_1 a_2 a_2^{\dagger} + a_2^{\dagger} a_2 a_1 a_1^{\dagger} + a_2^{\dagger 2} a_1^2 \rangle$$

$$\| \qquad 1 + a_2^{\dagger} a_2$$

$$= \langle a_1^{\dagger 2}\alpha^2 + a_1^{\dagger} a_1 (|\alpha|^2 + 1) + |\alpha|^2 a_1 a_1^{\dagger} + \alpha^{*2} a_1^2 \rangle$$

$$= 2|\alpha|^2 \langle x_\varphi^2 \rangle_S + \langle \underbrace{a_1^{\dagger} a_1}_{\hat{n}_1} \rangle_S \,, \qquad \Delta^2 A = 2|\alpha|^2 \Delta^2 x_\varphi + \langle \hat{n}_1 \rangle$$

$$\Delta^2 \tilde{x}_\varphi = \Delta^2 x_\varphi + \frac{\langle \hat{n}_1 \rangle_S}{2|\alpha|^2} \qquad \hat{n}_1 \;\; |\alpha|^2 \gg \langle n_1 \rangle_S \qquad \Delta^2 x_\varphi$$