8. Interaction of a two-level atom with quantized e-m field.

We use dipole approximation:

$$\hat{H} = \hat{H}_A + \hat{H}_L + H_I$$

$$\hat{H}_A = \sum_i \varepsilon_i |i\rangle\langle i| \quad , \quad \hat{H}_L = \sum_k \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\right), \quad \hat{H}_I = -\hat{\vec{d}} \cdot \hat{\vec{E}}(\vec{r}=0)$$

$$\begin{cases} \hat{\vec{d}} = \sum_{ij} \vec{d}_{ij} |i\rangle\langle j| \\ \hbar\nu_i \end{cases} , \quad \hat{\vec{E}}(\vec{r}=0) = \sum_k \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} \left(\hat{a}_k + \hat{a}_k^\dagger\right) \vec{e}_k$$

We move to interaction (Dirac) picture: $\hat{H}_0 = \hat{H}_A + \hat{H}_L$

$$\hat{H}_I^D(t) = e^{\frac{iH_0 t}{\hbar}} \hat{H}_I e^{-\frac{iH_0 t}{\hbar}}, \quad |\psi^D(t)\rangle = e^{\frac{iH_0 t}{\hbar}} |\psi(t)\rangle$$

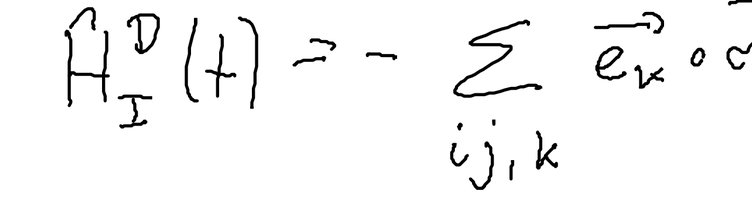
$$i\hbar \frac{d|\psi^D(t)\rangle}{dt} = H_I^D(t) |\psi^D(t)\rangle$$

In our case:

$$\hat{H}_I^D(t) = -\sum_{ij,k} \vec{e}_k \cdot \vec{d}_{ij} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} e^{i(\nu_i - \nu_j)t} |i\rangle\langle j| \otimes \left(\hat{a}_k e^{-i\omega_k t} + \hat{a}_k^\dagger e^{i\omega_k t}\right)$$

If we have discretized optical frequencies (in optical cavity) and some ω_{k_0} is close to $\nu_i - \nu_j$ $(i=g,\ j=e)$

In this case we restrict to only two atomic states $|g\rangle, |e\rangle$ and a single optical frequency $\omega := \omega_k$.



We switch to standing wave model:

$$\hat{\vec{E}} \simeq \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \left(\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right) \sin(kz)$$

$$H_I^D(t) = \sum_{i,j=g,e} \vec{e}_k \cdot \underline{\vec{d}_{ij}} \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \sin kz \, |i\rangle\langle j| e^{i(\nu_i - \nu_j)t} \otimes \left(\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right)$$

$$\vec{d}_{ii} = 0$$

$$\hat{\sigma}_+ = |e\rangle\langle g| \,, \quad \hat{\sigma}_- = |g\rangle\langle e| \,, \quad \nu = \nu_e - \nu_g$$

$$\lambda = \frac{\vec{e_k} \cdot \vec{d}_{eg}}{\hbar} \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \sin(kz)$$

$$\hat{H}_I^D(t) = \hbar\left(\lambda \hat{\sigma}_+ e^{i\nu t} + \lambda^* \hat{\sigma}_- e^{-i\nu t}\right)\left(\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right)$$

$$= \hbar\left(\lambda \hat{\sigma}_+ \hat{a}\, e^{i(\nu-\omega)t} + \lambda^* \hat{\sigma}_- \hat{a}^\dagger e^{-i(\nu-\omega)t}\right)$$

$\nu \approx \omega$

$\sim$

$\sim$

RWA

we just keep slowly varying terms

without loss of generality we can assume $\lambda = \lambda^*$ { by adding phase to e.g. $|g\rangle$ }

$$\hat{H}_I^D(t) = \hbar\lambda\left(\hat{\sigma}_+ \hat{a}\, e^{i\delta t} + \hat{\sigma}_- \hat{a}^\dagger e^{-i\delta t}\right)$$

(If we return to the Schrödinger picture

$$\hat{H} = \underbrace{\frac{1}{2}\hbar\nu\sigma_z}_{\text{we drop } \frac{1}{2}\hbar\omega} + \hbar\omega a^\dagger a + \hbar\lambda\left(\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger\right)$$

Jaynes-Cummings model

(dipole approximation + RWA)

Evolution in J-C model

We move both to the interaction picture

$$\frac{d|\psi^D(t)\rangle}{dt} = -\frac{i}{\hbar} H_I^D(t) |\psi^D(t)\rangle$$

general state:

$$|\psi^D(t)\rangle = |g\rangle \otimes \sum_n c_{g,n}|n\rangle + |e\rangle \otimes \sum_n c_{e,n}|n\rangle$$

$$\sum_n \dot{c}_{g,n}|g,n\rangle + \dot{c}_{e,n}|e,n\rangle = -i\lambda \left(\sum_n c_{g,n} e^{i\Delta t}\sqrt{n}\,|e,n-1\rangle + c_{e,n} e^{-i\Delta t}\sqrt{n+1}\,|g,n+1\rangle \right)$$

$$\begin{cases} \dot{c}_{e,n} = -i\lambda e^{i\Delta t} c_{g,n+1}\sqrt{n+1} \quad (*) \\ \dot{c}_{g,n} = -i\lambda e^{-i\Delta t} c_{e,n-1}\sqrt{n} \end{cases}$$

$$\begin{cases} \ddot{c}_{e,n} = \lambda\Delta e^{i\Delta t} c_{g,n+1}\sqrt{n+1} - i\lambda e^{i\Delta t}\dot{c}_{g,n+1}\sqrt{n+1} \\ \dot{c}_{g,n+1} = -i\lambda e^{-i\Delta t} c_{e,n}\sqrt{n+1} \end{cases}$$

$$\ddot{c}_{e,n} = i\Delta \dot{c}_{e,n} - \lambda^2(n+1)c_{e,n}$$

$$\ddot{c}_{e,n} - i\Delta \dot{c}_{e,n} + \frac{\Omega_R^2}{4} c_{e,n} = 0 \qquad \Omega_R^2 = 4\lambda^2(n+1)$$

Ω_R - Rabi frequency

$$x^2 - i\Delta x + \frac{\Omega_R^2}{4} = 0 \Rightarrow x_{n,\pm} = \frac{i}{2}\left(\Delta \pm \sqrt{\Delta^2 + \Omega_R^2}\right)$$

$$c_{e,n} = A_{n,+} e^{x_{n,+}t} + A_{n,-} e^{x_{n,-}t}$$

$$c_{g,n+1} = \frac{i}{\lambda}\frac{e^{-i\Delta t}}{\sqrt{n+1}}\left(x_{n,+}A_{n,+}e^{x_{n,+}t} + x_{n,-}A_{n,-}e^{x_{n,-}t}\right)$$

Initial state:

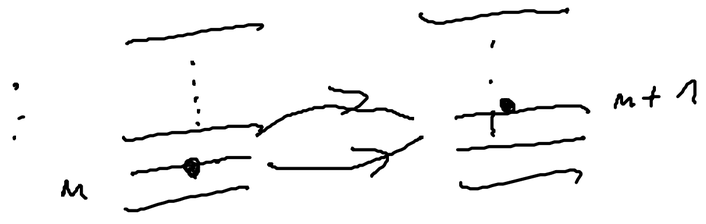
$$|\psi(0)\rangle = \sum_n c_{g,n}(0)|g,n\rangle + c_{e,n}(0)|e,n\rangle$$

$$\begin{cases} A_{n,+} + A_{n,-} = c_{e,n}(0) \\ \frac{i}{\lambda}\frac{1}{\sqrt{n+1}}\left(\chi_{n,+}A_{n,+} + \chi_{n,-}A_{n,-}\right) = c_{g,n+1}(0) \\ \vdots \end{cases}$$

$$\Omega = \sqrt{\Delta^2 + \Omega_R^2}$$

$$c_{e,n}(t) = e^{\frac{i\Delta t}{2}}\left\{ c_{e,n}(0)\left[\cos\left(\tfrac{1}{2}\Omega t\right) + \frac{i\Delta}{\Omega}\sin\left(\tfrac{1}{2}\Omega t\right)\right] - c_{g,n+1}(0)\,2i\sqrt{n+1}\,\frac{\lambda}{\Omega}\sin\left(\tfrac{1}{2}\Omega t\right)\right.$$

$$\left. c_{g,n+1}(t) = e^{-\frac{i\Delta t}{2}}\left\{ c_{e,n}(0)\,2i\,\frac{\lambda}{\Omega}\sqrt{n+1}\,\sin\left(\tfrac{1}{2}\Omega t\right) + c_{g,n+1}(0)\left[\cos\left(\tfrac{1}{2}\Omega t\right) + \frac{i\Delta}{\Omega}\sin\left(\tfrac{1}{2}\Omega t\right)\right]\right\}\right.$$



$$\Omega_R^2 = 4\lambda^2(n+1)$$

$$\Omega^2 = \Delta^2 + \Omega_R^2$$

Example $|\psi(0)\rangle = |e\rangle \otimes |n_0\rangle$

$$c_{e,n_0}(t) = e^{\frac{i\Delta t}{2}}\left[\cos\left(\tfrac{1}{2}\Omega t\right) + \frac{i\Delta}{\Omega}\sin\left(\tfrac{1}{2}\Omega t\right)\right]$$

$$c_{g,n_0+1}(t) = e^{-\frac{i\Delta t}{2}}\,2i\,\frac{\lambda}{\Omega}\sqrt{n+1}\,\sin\left(\tfrac{1}{2}\Omega t\right)$$

Atomic inversion parameter:

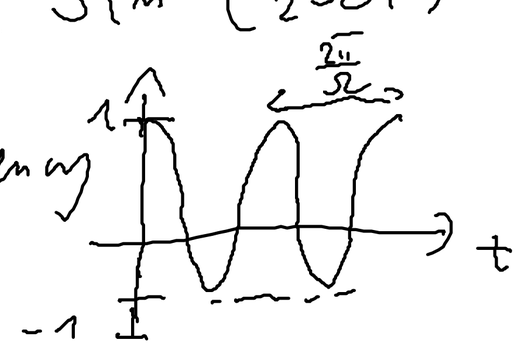
$$W(t) = \sum_n\left(|c_{e,n}|^2 - |c_{g,n}|^2\right) = \cos^2\left(\tfrac{1}{2}\Omega t\right) + \frac{\Delta^2}{\Omega^2}\sin^2\left(\tfrac{1}{2}\Omega t\right)$$

$$- \frac{4\lambda^2}{\Omega^2}(n+1)\sin^2\left(\tfrac{1}{2}\Omega t\right) = 1 - \left(1 + \frac{4\lambda^2(n+1) - \Delta^2}{\Omega^2}\right)\sin^2\left(\tfrac{1}{2}\Omega t\right) =$$

$$= 1 - \frac{\Omega^2 - \Delta^2 + \Omega_R^2}{\Omega^2}\sin^2\tfrac{1}{2}\Omega t = 1 - \frac{2\Omega_R^2}{\Omega^2}\sin^2\left(\tfrac{1}{2}\Omega t\right)$$

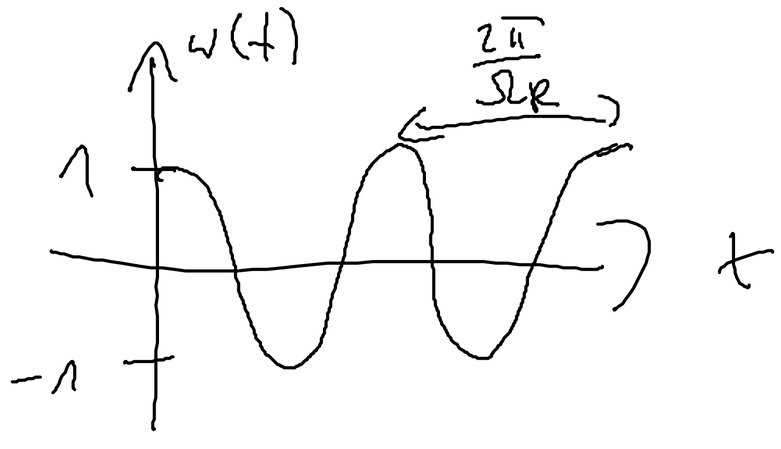
$$= 1 - \underbrace{\frac{\Omega_R^2}{\Omega^2}}\left(1 - \cos\underbrace{\Omega t}\right)$$

generalized Rabi frequency



$\Delta = 0 \qquad \nu = \omega$

$w(t) = \qquad \cos(\Omega_R t)$



$\Omega_R = 2\lambda\sqrt{n+1}$

{ 2014 Nobel Prize
Wineland, Haroche

What is "quantum" here ?

{ In semi-classical model (where light is classical)
we would also have Rabi oscillations $\Omega_R = \frac{|E| d}{\hbar}$

Here even if we have vaccum $n=0$ we have oscillations. In semi-classical model for $|E|=0$
we would have no oscillations.

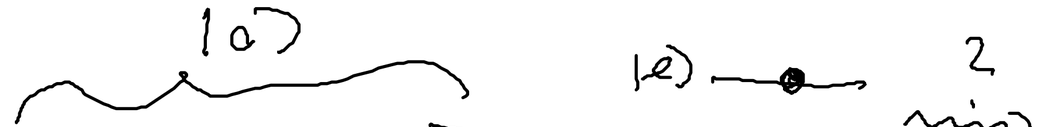
Example $|\psi(0)\rangle = |e\rangle \otimes |\alpha\rangle$ ← coherent state

Spontaneous emission in free space

Two level atom + all the modes in free space.

$$H_I^D(t) = \hbar \sum_{\vec{k}} \left(\lambda_{\vec{k}}^* \hat{\sigma}_+ \hat{a}_{\vec{k}} e^{i(\nu - \omega_k)t} + h.c \right)$$

$$\lambda_{\vec{k}} = \frac{\vec{e}_k \cdot \vec{d}_{eg}}{\hbar} \sqrt{\frac{\hbar \omega_k}{2 \varepsilon_0 V}}$$



Initially $|\psi(0)\rangle = |e\rangle \otimes |0,0,\dots\dots 0\rangle$, how fast the atom will decay?

General state after time t will have a form:

$$|\psi^D(t)\rangle = c_e(t)\,|e\rangle\otimes|0\rangle + \sum_{\vec{k}} c_{g,\vec{k}}(t)\,|g\rangle \otimes |1\rangle_{\vec{k}}$$

$|1\rangle_{\vec{k}}$ = more $\vec{k}$: $|0,0\dots,\overset{\vec{k}}{1},0\dots 0\rangle$

$$\begin{cases} c_e(0) = 1 \\ c_{g,\vec{k}}(0) = 0 \end{cases}$$

$$|\dot{\psi}^D(t)\rangle = -\frac{i}{\hbar} H_I^D(t)\,|\psi^D(t)\rangle =$$

$$= -i\,c_e(t) \sum_{\vec{k}} \lambda_{\vec{k}}\, e^{-i(\nu-\omega_k)t}\,|g\rangle\otimes|1\rangle_{\vec{k}} +$$

$$-i \sum_{\vec{k}} \lambda_{\vec{k}}^*\, c_{g,\vec{k}}(t)\, e^{i(\nu-\omega_k)t}\,|e\rangle\otimes|0\rangle$$

$$\Downarrow$$

$$\begin{cases} \dot{c}_e(t) = -i \sum_{\vec{k}} \lambda_{\vec{k}}^* e^{i(\nu-\omega_k)t}\, c_{g,\vec{k}}(t) \\ \dot{c}_{g,\vec{k}}(t) = -i\,\lambda_{\vec{k}}\, e^{-i(\nu-\omega_k)t}\, c_e(t) \end{cases} \Rightarrow c_{g,\vec{k}}(t) = -i\lambda_{\vec{k}} \int_0^t dt'\, e^{-i(\nu-\omega_k)t'}\, c_e(t')$$

$$\dot{c}_e(t) = -\sum_{\vec{k}} |\lambda_k|^2 \int_0^t dt'\, e^{i(\nu-\omega_k)(t-t')}\, c_e(t')$$

We move to continuum limit:

$$C_e(t) = \sum_{\vec{k}} \frac{2|d_{eg}|^2}{3(2\pi)^2 \hbar \varepsilon_0} \frac{V}{(2\pi)^3} \int d^3k \qquad \int d^3k = \int dk\, k^2 \int d\theta \sin\theta \int d\varphi$$

↑ two polarizations

$$\dot{C}_e(t) = -\frac{2V}{(2\pi)^3} \int dk\, k^2 \int d\theta \sin\theta \int d\varphi \; \frac{\omega_k}{2\varepsilon_0 V \cdot \hbar} \underbrace{\left(\vec{e}_k \cdot \vec{d}_{eg}\right)^2}_{|d_{eg}|^2 \cos^2\theta} \int_0^t dt' e^{i(\nu-\omega_k)(t-t')} C_e(t')$$

$$\left\{ \int_0^{\pi} d\theta \sin\theta \cos^2\theta = \frac{2}{3} \right. \qquad \omega = k \cdot c$$

$$\dot{C}_e(t) = \frac{-2|d|^2}{3(2\pi)^2 \varepsilon_0 \hbar c^3} \int_0^{\infty} d\omega\, \omega^3 \int_0^t e^{i(\nu-\omega)(t-t')} C_e(t')$$

We make Markovianity assumption (no memory in the environment)

and we assume that $C_e(t)$ varies slowly (compared with ν)

$$A = \int_0^{\infty} d\omega\, \omega^3 \int_0^t e^{i(\nu-\omega)(t-t')} C_e(t')\, dt' \simeq \left\{ \begin{array}{l} \text{Wigner-} \\ \text{-Weisskopf} \\ \text{approximation} \end{array} \right.$$

$C_e(t')$: varies slowly

$$\simeq C(t) \int_0^{\infty} d\omega\, \omega^3 \int_{-\infty}^{t} e^{i(\nu-\omega)(t-t')} dt' = \quad \tau = t - t'$$

$$= C(t) \int_0^{\infty} d\omega\, \omega^3 \underbrace{\int_0^{\infty} e^{i(\nu-\omega)\tau} d\tau}_{I}$$

$$I = \lim_{\varepsilon \to 0} \int_0^{\infty} e^{i(\nu-\omega)\tau - \varepsilon\tau} d\tau = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon - i(\nu-\omega)} =$$

$$= \lim_{\varepsilon \to 0} \left[\frac{\varepsilon}{\varepsilon^2 + (\nu-\omega)^2} + \frac{i(\nu-\omega)}{\varepsilon^2 + (\nu-\omega)^2} \right] = \pi \delta(\nu-\omega) + \underbrace{i P\left(\frac{1}{\nu-\omega}\right)}_{\text{not very important.}}$$

$$\dot{c}_e(t) = \frac{-2|d|^2}{3(2\pi)^2 \varepsilon_0 \hbar c^3} \int_0^\infty d\omega\, \omega^3 \left(\pi\, \delta(\nu-\omega) + \text{imaginary term} \right) \cdot c_e(t)$$

If we just care about the rate of spontaneous emission we just care about: $|c_e(t)|^2$

In general: $\frac{dc}{dt} = a \cdot c + ib \cdot c$, $\frac{d|c|^2}{dt} = \frac{d}{dt}(c c^*) = c\frac{dc^*}{dt} + \frac{dc}{dt} \cdot c^* =$

$a, b \in \mathbb{R}$

$$= c(ac^* - ibc^*) + c^*(ac + ibc) = \underline{\underline{2a|c|^2}}$$

So imaginary term will not influence the rate but it will effectively add some $e^{i\beta \cdot t}$ factor to $c_e(t)$.

effectively this will result in a Lamb shift of transition frequency $\nu \to \nu - \beta$.

$$\frac{d|c_e(t)|^2}{dt} = -2 \cdot \underbrace{\frac{2|d|^2 \pi \nu^3}{3(2\pi)^2 \varepsilon_0 \hbar c^3}}_{\Gamma = \frac{|d|^2 \nu^3}{3\pi \varepsilon_0 \hbar c^3}} |c_e(t)|^2$$

$$|c_e(t)|^2 = e^{-\Gamma t}$$

If we neglect the Lamb shift: $c_e(t) = e^{-\frac{\Gamma}{2} \cdot t}$

And we can write the state of light:

$$c_{g,\vec{k}}(t) = -i\lambda_k \int_0^t dt' e^{-(\nu-\omega_k)t' - \frac{\Gamma}{2}t'} = \lambda_k \frac{1 - e^{-i(\nu-\omega_k)t - \frac{\Gamma}{2}t}}{(\nu-\omega_k) + i\frac{\Gamma}{2}}$$

$t \to \infty$ $\longrightarrow c_{\vec{k}} = \frac{\lambda_k}{(\nu-\omega_k) + i\frac{\Gamma}{2}}$, $|\psi(t)\rangle \overset{t\to\infty}{\approx} |g\rangle \otimes \sum_{\vec{k}} c_{\vec{k}} |1\rangle_{\vec{k}}$, $|c_k|^2 = \frac{|\lambda_k|^2}{(\nu-\omega_k)^2 + \frac{\Gamma^2}{4}}$