II. Interactions of q.light with matter

6. Quantum electromagnetic field + nonrelativistic quantum point charge.

Lagrangian: $L(\vec{q}, \dot{\vec{q}}, A_\mu, \partial_\nu A_\mu)$

$$L = \int d^3r\, \mathcal{L} + \frac{m\dot{q}^2}{2}$$

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu}F_{\mu\nu} - j^\mu A_\mu$$

↑ Lagrangian variables

$$\begin{cases} j^\mu = [\rho c, \vec{j}] \\ \rho(\vec{r}) = e\,\delta^{(3)}(\vec{r} - \vec{q}) \\ \vec{j} = e\dot{\vec{q}}\,\delta^{(3)}(\vec{r} - \vec{q}) \end{cases}$$

↑ Lagrangian variable

We use the Coulomb gauge $\vec{\nabla}\vec{A} = 0$

Without charges we could additionally put $\varphi = 0$

In general $\vec{A} = \vec{A}_\perp + \vec{A}_{\parallel}$ $\quad \{ \nabla\vec{A}_\perp = 0, \quad \vec{\nabla}\times\vec{A}_{\parallel} = 0$

Gauge transformation: $\vec{A}_c = \vec{A} - \vec{\nabla}\chi$ $\quad$ In Coulomb Gauge we choose χ such that $\vec{\nabla}\vec{A}_c = 0$ we remove $\vec{A}_{\parallel}$ from $\vec{A}$

$$\varphi_c = \varphi + \frac{\partial\chi}{\partial t}$$

Electric field $\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \nabla\varphi = -\frac{\partial\vec{A}_c}{\partial t} - \vec{\nabla}\varphi_c$

$\vec{E}_\perp$ $\quad$ $\vec{E}_{\parallel}$

Without charges: $\vec{E} = \vec{E}_\perp$

With charges we will have additionally $\vec{E}_{\parallel} = -\vec{\nabla}\varphi_c$,

φ_c - will be determined by charges

$\{\cdot,\cdot\} \to \frac{1}{i\hbar}[\cdot,\cdot]$, $\quad \hat{A}_i, \hat{\Pi}_i, \hat{q}_i, \hat{p}_i$

Quantization procedure:

$[\hat{q}_i, \hat{p}_j] = i\hbar\,\delta_{ij}$

$[\hat{A}_i(\vec{r}), \hat{\Pi}_j(\vec{r}')] = i\hbar\,\delta^{tr}_{ij}(\vec{r} - \vec{r}')$

other commutators are zero

The result of quantization procedure:

$$H = \frac{1}{2m}\left(\hat{\vec{p}} - e\hat{\vec{A}}(\hat{\vec{q}})\right)^2 + \underbrace{\frac{1}{2}\int d^3r\left[\varepsilon_0 \hat{\vec{E}}_\perp^2 + \frac{1}{\mu_0}\left(\vec{\nabla}\times\hat{\vec{A}}\right)^2\right]}_{\text{the same as in free theory:}}$$

$$-\varepsilon_0\left[\hat{A}_{i\perp}(\vec{r}), \hat{E}_{j\perp}(\vec{r}')\right] = i\hbar\,\delta_{ij}^{tr}(\vec{r}-\vec{r}')$$

$$+\frac{1}{2}\varepsilon_0\int \hat{\vec{E}}_\parallel^2\, d^3r$$

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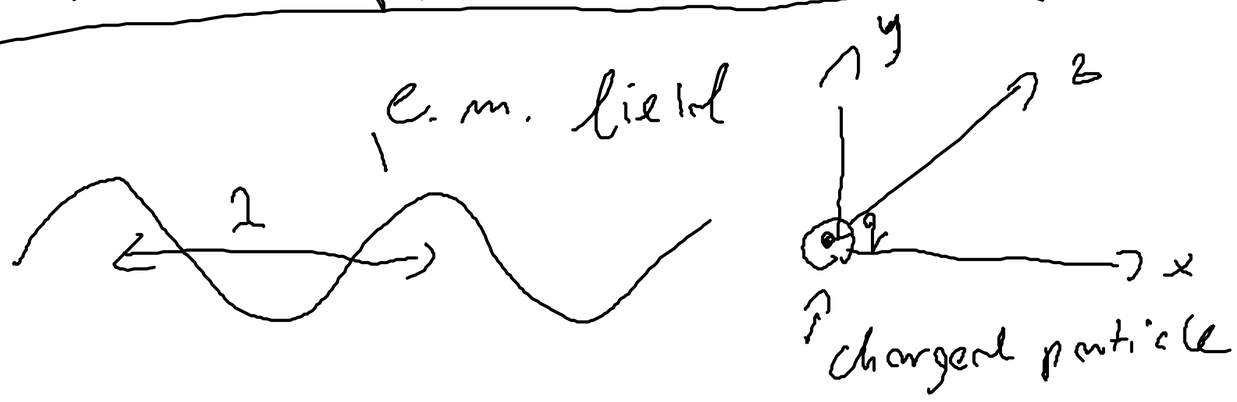
Dipole approximation



$$|\vec{q}| \ll \lambda$$

$$\hat{\vec{A}}(\hat{\vec{r}}) \approx \hat{\vec{A}}(0)$$

We will perform a unitary transformation:

$$|\psi'\rangle = \mathcal{U}|\psi\rangle, \qquad \mathcal{U}^+\hat{A} = \mathcal{U} = \hat{A}'$$

we take: $\mathcal{U} = e^{\frac{i}{\hbar}e\hat{\vec{q}}\cdot\hat{\vec{A}}(0)}$

$$\begin{cases} \hat{\mathcal{U}}^+\hat{\vec{q}}\,\hat{\mathcal{U}} = \hat{\vec{q}} \\ \mathcal{U}^+\hat{\vec{p}}\,\mathcal{U} = \hat{\vec{p}} + \left[-\frac{i}{\hbar}e\hat{\vec{q}}\cdot\vec{A}, \hat{\vec{p}}\right] = \hat{\vec{p}} + e\hat{\vec{A}}(0) \\ \mathcal{U}^+\hat{\vec{A}}\,\mathcal{U} = \hat{\vec{A}} \\ \mathcal{U}^+\hat{E}_{i\perp}(\vec{r})\mathcal{U} = \hat{E}_{i\perp}(\vec{r}) + \left[-\frac{i}{\hbar}e\hat{\vec{q}}\,\hat{\vec{A}}(0), \hat{E}_{i\perp}(\vec{r})\right] = \hat{E}_{i\perp}(\vec{r}) - \frac{e}{\varepsilon_0}\delta_{ij}^{tr}(\vec{r})\cdot\hat{q}_j \end{cases}$$

$$H' = U^\dagger H U = \frac{\vec{p}^2}{2m} + \frac{1}{2}\varepsilon_0 \int d^3r\, \vec{E}_{\parallel}^2 +$$

$$+ \frac{1}{2}\int d^3r\, \varepsilon_0 \left(\vec{E}^\perp - \frac{e}{\varepsilon_0}\delta^\perp(\vec{r})\,\vec{q} \right)^2 + \frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{A}\right)^2 \Bigg]$$

$$\int d^3r\, \vec{E}^\perp(\vec{r})\cdot\delta^\perp(\vec{r})\,\vec{q} = \int d^3r\, \vec{E}^\perp(\vec{r})\,\delta^{(3)}(\vec{r})\,\vec{q} =$$

because we integrate with $\vec{E}^\perp$ for which $\nabla\vec{E}^\perp = 0$

$$= \vec{E}(0)\cdot\vec{q}$$

$$\int d^3r \left(\delta^T(\vec{r})\,\vec{q}\right)^2 = \int d^3r \sum_{ijk} \delta^T_{ik}(\vec{r})\,q_k\,\delta^\perp_{ij}(\vec{r})\,q_j =$$

$$= \sum_{ij} q_i\,\delta^\perp_{ij}(0)\,q_j = \vec{q}^{\,T}\delta^\perp(0)\,\vec{q}$$

$$H' = \frac{\vec{p}^2}{2m} + \frac{1}{2}\varepsilon_0\int d^3r\, \vec{E}_{\parallel}^2 \underbrace{- e\,\vec{q}\,\vec{E}^\perp(0)}_{\text{Dipole interaction Hamiltonian}} + \underbrace{\frac{e^2}{2\varepsilon_0}\,\vec{q}^{\,T}\delta^\perp(0)\,\vec{q}}_{\text{relevant for Lamb shift}}$$

$$+ \frac{1}{2}\int d^3r \left(\varepsilon_0\, \vec{E}^{\perp 2} + \frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{A}\right)^2 \right)$$

For the time being we will just use.

$$H \approx \sum_\lambda \int d^3k\, \hbar\omega \left(\hat{a}^\dagger(k,\lambda)\,\hat{a}(k,\lambda) + \frac{1}{2} \right) - e\,\vec{q}\,\vec{E}^\perp$$

$$\vec{E}^\perp = \int d^3k \sqrt{\frac{\hbar\omega_k}{(2\pi)^3 2\varepsilon_0}} \sum_\alpha \vec{e}(k,\lambda)\left(\hat{a}(k,\lambda)\,e^{i\vec{k}\vec{r}-\omega t} + \hat{a}^\dagger(k,\lambda)\,e^{-i\vec{k}\vec{r}+\omega t} \right)$$