4. Classical & non-classical states of light

Quantum E-M field from classical sources

Lagrangian: $\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - \underbrace{j^\mu A_\mu}_{\mathcal{L}_{classical}}$ $\qquad j^\mu = [c\rho, \vec{j}]$

Equation of motion: $\Box A^\mu = \mu_0 j^\mu$

Note: $\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_t A^\mu)}$ - the same as in no-sources case

The only difference in quantization procedure:

$$\hat{\mathcal{H}} = \sum_\mu \hat{\Pi}^\mu \partial_t A_\mu - \mathcal{L} = \hat{\mathcal{H}}_0 + j^\mu \hat{A}_\mu$$

$$\hat{H} = \int d^3r\, \hat{\mathcal{H}} = \hat{H}_0 + \underbrace{\int d^3\vec{r}\; j^\mu \hat{A}_\mu}_{H_{int}}$$

{ In Coulomb gauge $A^\circ = \frac{\varphi}{c}$ is still classical and is determined by ρ - classical charge density.

{ In what follows we set $\rho = 0$

$$H_{int} = \int d^3\vec{r}\; \vec{j} \cdot \vec{A}.$$

In no-sources case we worked in the Heisenberg picture and found the solution: (for discrete modes)

$$\hat{\vec{A}}(\vec{r},t) = i\mathcal{A} \sum_k \frac{1}{\sqrt{\omega_k}} \vec{e}_k \left(\hat{a}_k e^{i(\vec{k}\vec{r} - \omega_k t)} - \hat{a}_k^\dagger e^{-i(\vec{k}\vec{r} - \omega_k t)} \right)$$

$$\mathcal{A} = \sqrt{\frac{\hbar}{2\varepsilon_0 V}}$$

To see how presence of $\vec{j}$ changes the quantum states of E-M field we use the interaction picture (Dirac picture)

$$\hat{\vec{A}}(t) = e^{\frac{iH_0 t}{\hbar}} \hat{\vec{A}}(0)\, e^{-\frac{iH_0 t}{\hbar}} \qquad |\psi(t)\rangle_D := e^{\frac{iH_0 t}{\hbar}} |\psi(t)\rangle_S$$

$$\frac{d|\psi(t)\rangle_D}{dt} = -\frac{i}{\hbar} \hat{H}_{int}(t) |\psi(t)\rangle_D$$

Assume: $|\psi(t=0)\rangle_D = |0\rangle$ vaccum state

what is state of E-M field at time t?

$$|\psi(t)\rangle_D = \mathcal{T}\left[e^{-\frac{i}{\hbar}\int_0^t H_{int}(t')\,dt'} \right] |0\rangle = \left\{ \hat{H}_{int}(t) = \int d^3\vec{r}\; \vec{j}(\vec{r},t)\cdot\hat{\vec{A}}(\vec{r},t) \right.$$

$\left.\right\}$ because $[H_{int}(t), H_{int}(t')] \neq 0$

$$= \mathcal{T}\left[\sum_k \frac{A}{\hbar} \int_0^t dt' \int \frac{d^3\vec{r}}{\sqrt{\omega_k}}\, \vec{j}(\vec{r},t)\cdot\vec{e}_k \left(\hat{a}_k e^{i(\vec{k}\vec{r}-\omega_k t)} - \hat{a}_k^\dagger e^{-i(\vec{k}\vec{r}-\omega_k t)} \right) \right] |0\rangle$$

$$= \mathcal{T}\left[\prod_k e^{\alpha_k(t)\hat{a}_k^\dagger - \alpha_k^*(t)\hat{a}_k} \right] |0\rangle =$$

$$\left\{ \alpha_k(t) = -\frac{1}{\sqrt{2\varepsilon_0 V \hbar \omega_k}} \int_C dt' \int d^3\vec{r}\; \vec{j}(\vec{r},t')\cdot\vec{e}_k\, e^{-i(\vec{k}\vec{r}-\omega_k t')} \right.$$

$$\left\{ e^{\frac{i}{\hbar}\int_t H_{int}(t_1) + \frac{i}{\hbar}\int_t H_{int}(t_2)} \overset{BCH}{=} e^{\frac{i}{\hbar}\int_t H_{int}(t_2)} e^{\frac{i}{\hbar}\int_t H_{int}(t_1)} \cdot e^{-\frac{1}{2}\underbrace{\left[\frac{i}{\hbar}\int_t H_{int}(t_2), \frac{i}{\hbar}\int_t H_{int}(t_1)\right]}_{\text{not an operator just a function}}} \right.$$

$$H_{int}(t_1) = f(t_1)\hat{a} + f^*(t_1)\hat{a}^\dagger + \dots$$

up to some multiplying function $e^{i\xi(t)}$

$$= \prod_k e^{\alpha_k(t)\hat{a}_k^\dagger - \alpha_k^*(t)\hat{a}_k} |0\rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \dots$$

product of coherent states

$$= \bigotimes_k |\alpha_k(t)\rangle$$

The Vaccum becomes ~ coherent state due to classical currents.

Moving back to Schrödinger picture:

$$|\psi(t)\rangle_S = e^{-\frac{i}{\hbar}H_0 t}\,|\psi(t)\rangle_D =$$

$$= \bigotimes_k |\, e^{-i\omega_k t}\,\alpha_k(t)\rangle$$

Classical states of light

Definition:
A k-mode state $\hat{\rho}$ of light is classical

$$\Longleftrightarrow \hat{\rho} = \int d^2\alpha_1 \dots d^2\alpha_k\, P(\alpha_1,\dots,\alpha_k)\,|\alpha_1\rangle\langle\alpha_1| \otimes |\alpha_2\rangle\langle\alpha_2| \otimes \dots \otimes |\alpha_k\rangle\langle\alpha_k|$$

$$P(\alpha_1,\dots,\alpha_k) \geq 0$$

probabilistic mixture of coherent states

All other states are called non-classical.

Squeezed states

Focus on a single mode

$\hat{a}, \hat{a}^\dagger$ ⟶ introduce $\hat{q} := \frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}$, $\hat{p} := \frac{\hat{a}-\hat{a}^\dagger}{i\sqrt{2}}$

$\hat{q}, \hat{p}$ – quadrature operators

For a single mode: (think of plane wave) $[\hat{q}, \hat{p}] = i$

$$\hat{\vec{E}} = \mathcal{E}_0\left(\hat{a}\,e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right)\sin(kz)\,\hat{e}_x =$$

$$= \sqrt{2}\,\mathcal{E}_0\left(\hat{q}\cos\omega t + \hat{p}\sin\omega t\right)\sin(kz)\,\hat{e}_x$$

Consider a coherent state $|\alpha\rangle$

$$\langle\alpha|\hat{q}|\alpha\rangle = \langle\alpha|\frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}|\alpha\rangle = \sqrt{2}\,\mathrm{Re}\,\alpha$$

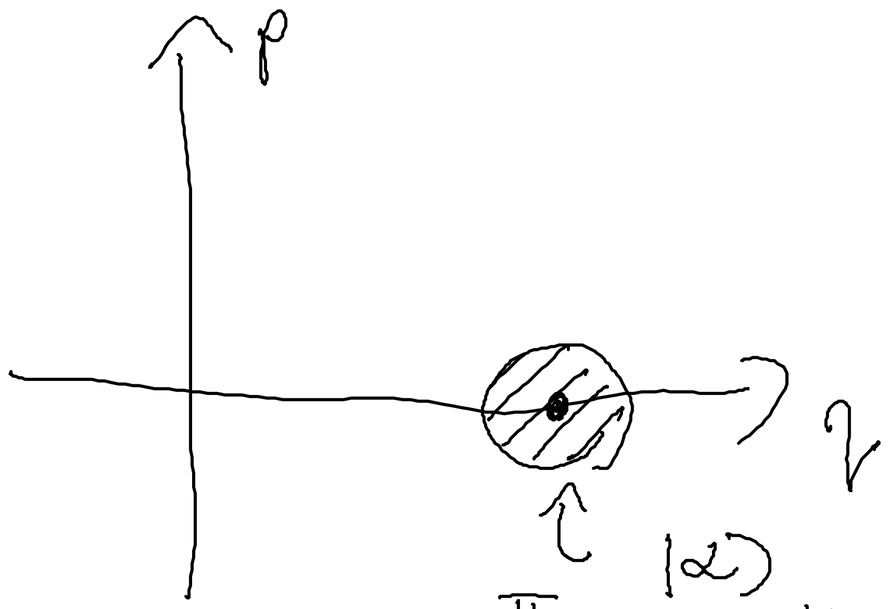
$$\langle\alpha|\hat{p}|\alpha\rangle = \sqrt{2}\,\mathrm{Im}\,\alpha$$

$$\langle\alpha|\hat{q}^2|\alpha\rangle = 2(\mathrm{Re}\,\alpha)^2 + \frac{1}{2}$$

$$\Delta^2 q = \langle\alpha|\hat{q}^2|\alpha\rangle - \langle\alpha|q|\alpha\rangle^2 = \frac{1}{2}$$

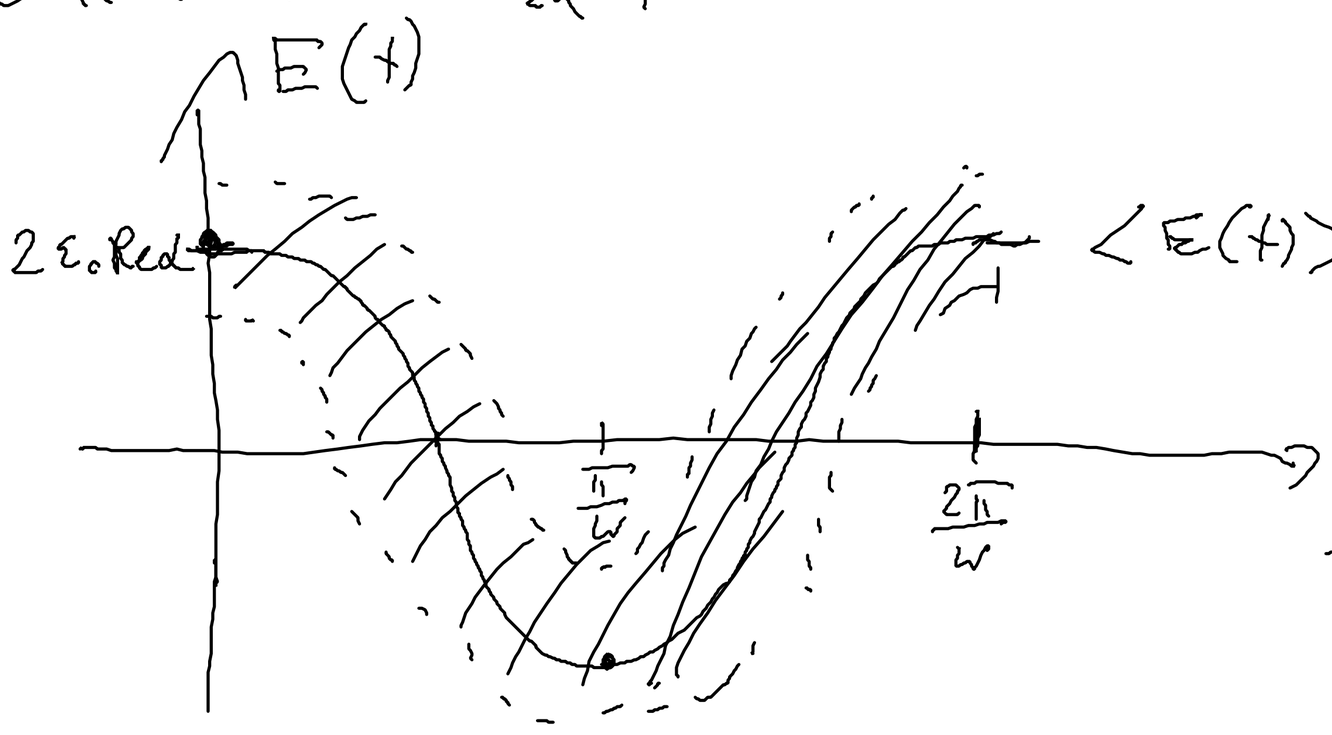
$$\vdots$$

$$\Delta^2 p = \frac{1}{2}$$



$$\hat{E}(t) = \sqrt{2}\,\mathcal{E}_0\left(\hat{q}\cos\omega t + \hat{p}\sin\omega t\right)$$

we look at $z = \frac{\pi}{2k}$, $\sin kz = 1$



$$\langle E(t)\rangle = \langle\alpha|\hat{E}|\alpha\rangle$$

$\Delta^2 E(t)$ - will not depend on time.

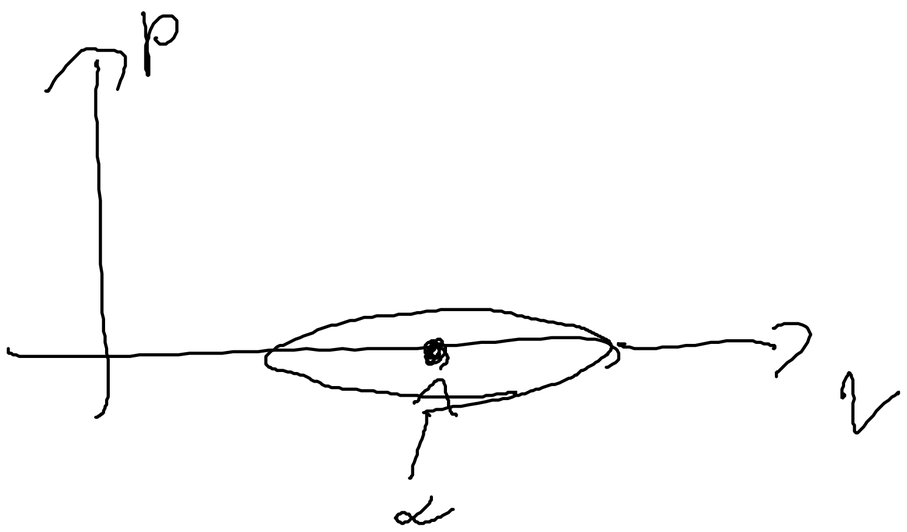
$$\hat{E} = \sqrt{2}\,\mathcal{E}_0\left(\hat{q}\cos\omega t + \hat{p}\sin\omega t\right)\sin(kz)\,\hat{e}_x$$

We know that $\Delta^2 q$ and $\Delta^2 p$ should obey Heisenberg uncertainty relation

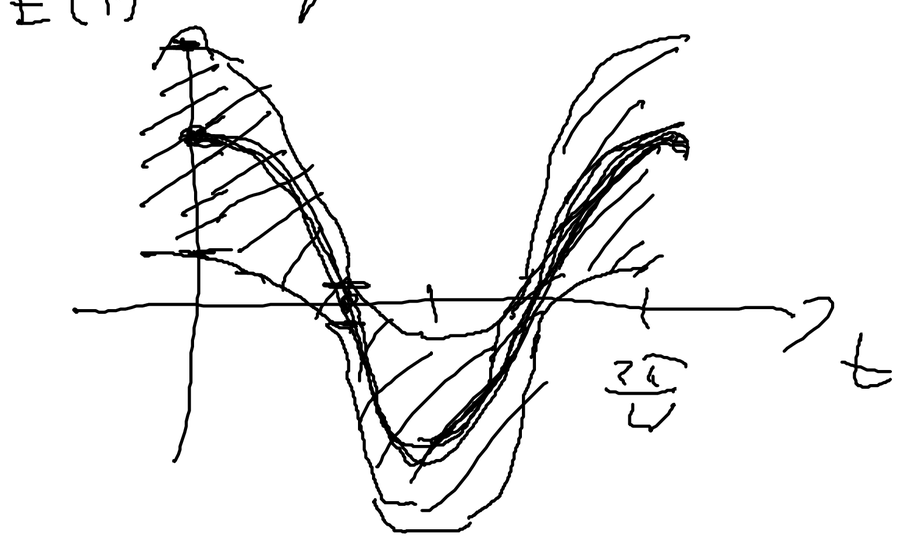
$$\Delta^2 q\,\Delta^2 p \geq \frac{1}{4}$$

We can imagine states for which $\Delta^2 p = \frac{1}{2}e^{-2r}$

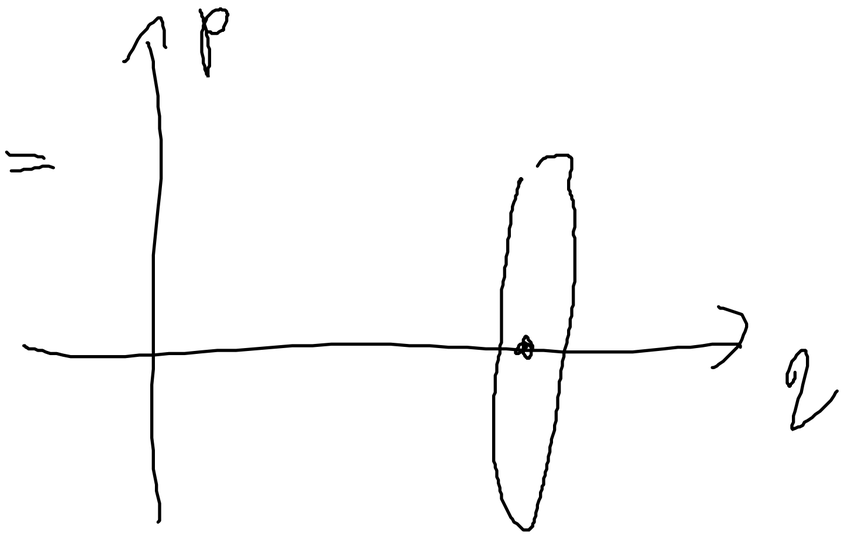
$|\alpha, r\rangle =$

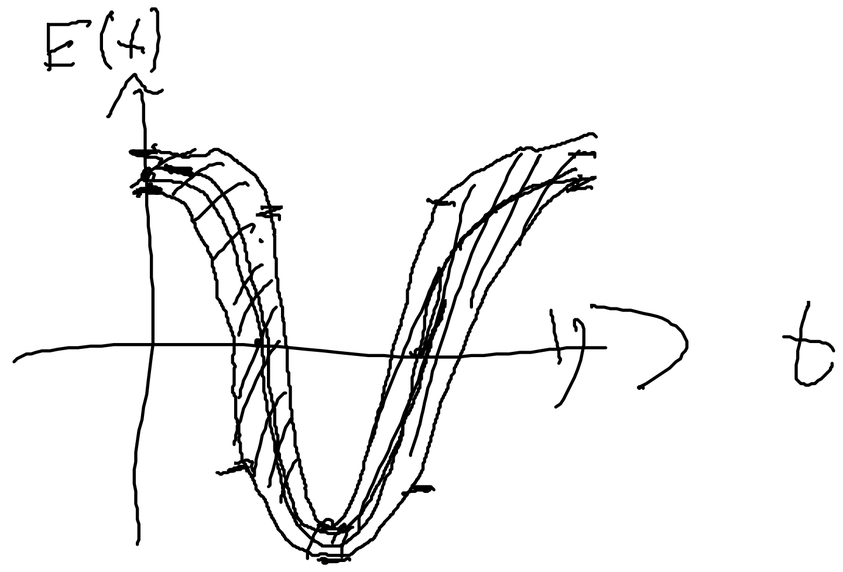


$\Delta^2 q = \frac{1}{2}e^{+2r}$



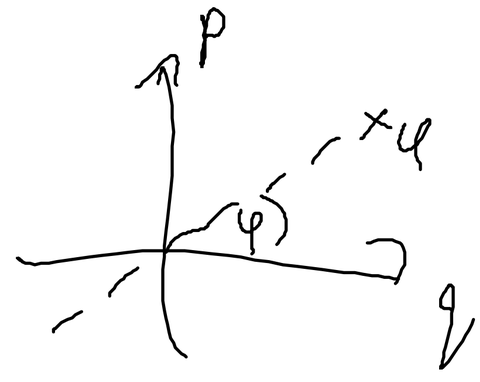
$|\alpha_1 - r\rangle =$





We can introduce a generalized quadrature operator

$$\hat{x}_\varphi = \cos\varphi\ \hat{q} + \sin\varphi\ \hat{p}$$



Definition:
We call a state squeezed if for some φ $\Delta^2 x_\varphi < \frac{1}{2}$

Squeezed coherent States

$$|\alpha, r, \theta\rangle := \hat{D}(\alpha)\ \hat{S}(r,\theta)\,|0\rangle$$

$\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$ displacement operator

$\hat{S}(r,\theta) = e^{\frac{1}{2}(\zeta^*\hat{a}^2 - \zeta\hat{a}^{\dagger 2})}$, $\zeta = r e^{i\theta}$ squeezing operator

Compute:

$$\hat{S}^\dagger \hat{a}\ \hat{S} \overset{\text{def}}{=} \hat{a}\cosh(r) - \hat{a}^\dagger \sinh(r)\cdot e^{i\theta}$$

$$S^\dagger \hat{a}^\dagger \hat{S} = \hat{a}^\dagger \cosh(r) - \hat{a}\sinh(r)\, e^{-i\theta}$$

$$S^\dagger \hat{q}\, S \overset{\text{def}}{=} \hat{q}(\cosh r - \sinh r\cos\theta) - \hat{p}\sinh r\sin\theta$$

$$S^\dagger \hat{p}\, S = -\hat{q}\sinh r\sin\theta + \hat{p}(\cosh r + \sinh r\cos\theta)$$

$$\begin{bmatrix} \hat{q}_1 \\ \hat{p}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} \cosh r - \sinh r \cos\theta, & -\sinh r \sin\theta \\ -\sinh r \sin\theta, & \cosh r + \sinh r \cos\theta \end{bmatrix}}_{} \begin{bmatrix} \hat{q} \\ \hat{p} \end{bmatrix}$$

eigenvalues $\lambda_+ = e^r$, $\lambda_- = e^{-r}$

it means:

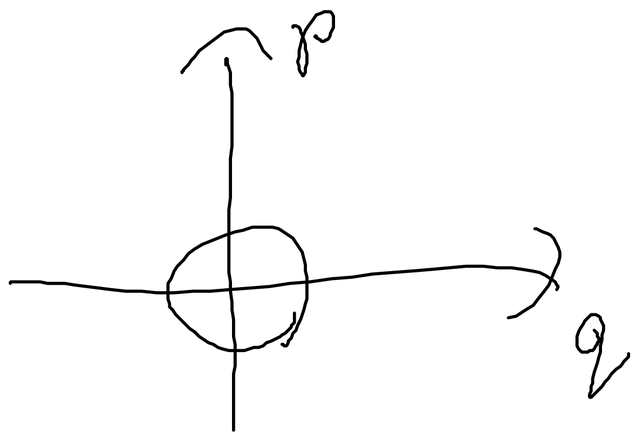
$$x_+ = \begin{bmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{bmatrix}, \quad x_- = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix}$$

$\hat{x}_+' = e^r \hat{x}_+$ — antisqueezing

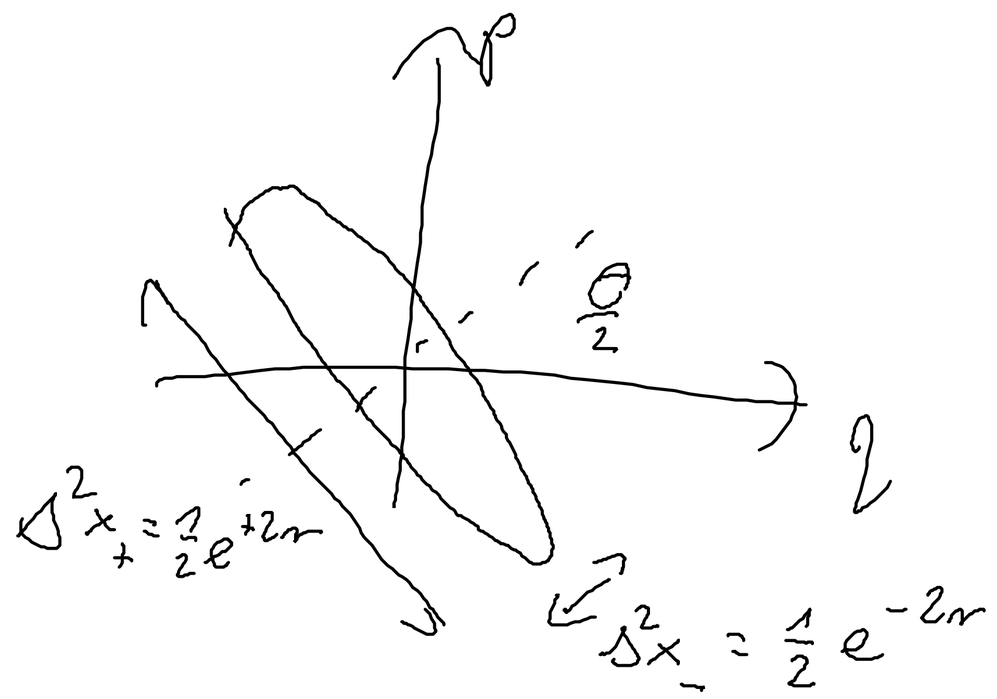
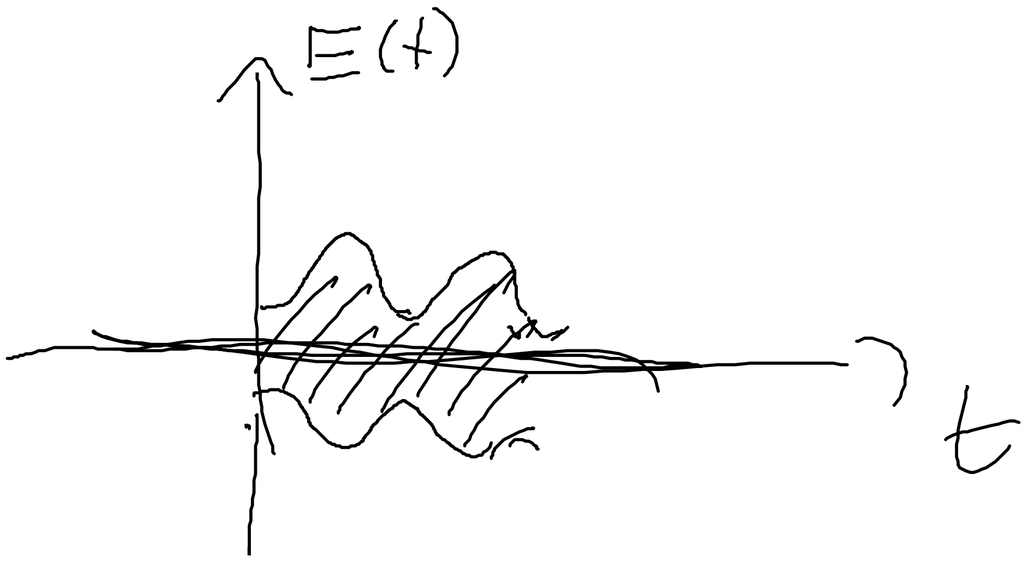
$\hat{x}_-' = e^{-r} \hat{x}_-$ — squeezing

It means that in the Schrödinger picture:

$|0\rangle =$



$\Rightarrow |0, r, \theta\rangle = \hat{S}(r,\theta)|0\rangle$



$\hat{S}(r,\theta)|0\rangle$ — squeezed vacuum state