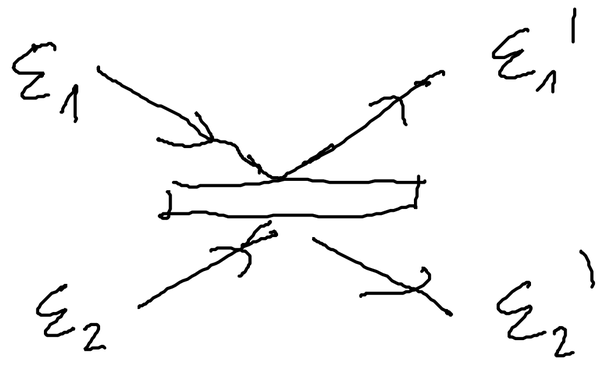
3. Linear Optical Networks

Mirrors, Beamsplitters, Interferometers ... + Detectors

- Beamsplitter. (classically)



$$\begin{bmatrix} \mathcal{E}_1' \\ \mathcal{E}_2' \end{bmatrix} = \begin{bmatrix} r_1 & t_2 \\ t_1 & r_2 \end{bmatrix} \begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{bmatrix}$$

B

energy conservation: $|\mathcal{E}_1|^2 + |\mathcal{E}_2|^2 = |\mathcal{E}_1'|^2 + |\mathcal{E}_2'|^2$

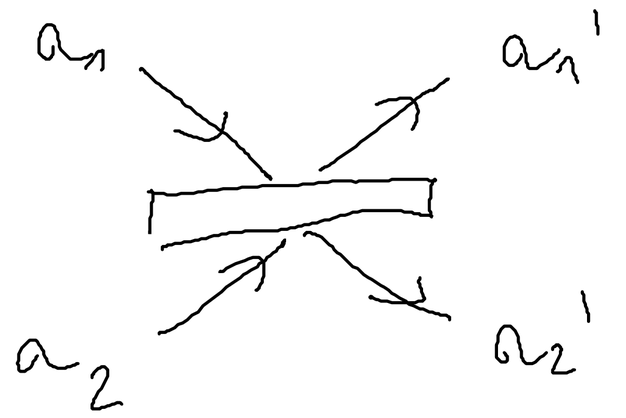
$\Downarrow$

B – unitary

e.g. 50:50 beamsplitter $B_{50/50} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

general $B = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}$

- Beamsplitter (quantum optics)



$$\begin{bmatrix} \hat{a}_1' \\ \hat{a}_2' \end{bmatrix} = B \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix}$$

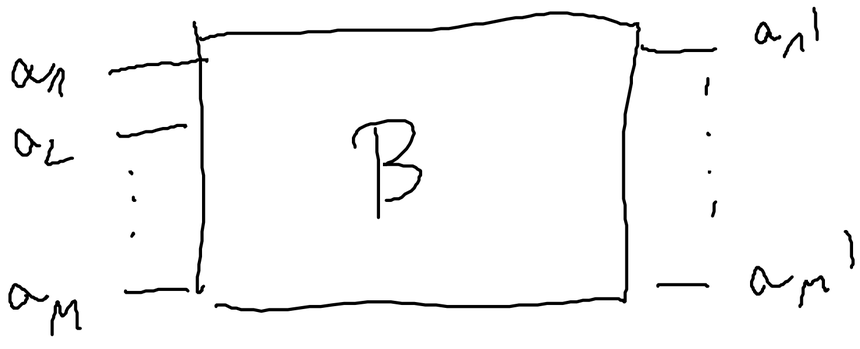
(Heisenberg picture)

instead of computing: $\langle\psi| f(\vec{a}, \vec{a}^\dagger) |\psi\rangle$ $\xrightarrow{\text{presence of beamsplitter}}$ $\langle\psi| f(\vec{a}', \vec{a}'^\dagger) |\psi\rangle$

$\hat{a}_j$ – input modes $\hat{a}_j'$ – output modes

Evolution $\equiv$ transformation of modes.

In general: (linear optical transformation (passive))



$$\vec{a}' = B\,\vec{a}$$

We want to describe it as a unitary transformation in the Hilbert space:

$$\vec{a}' = B\,\vec{a} = \begin{bmatrix} \hat{U}_B^\dagger \hat{a}_1 \hat{U}_B \\ \vdots \\ \hat{U}_B^\dagger \hat{a}_M \hat{U}_B \end{bmatrix} = \hat{U}_B^\dagger\, \vec{\hat{a}}\, \hat{U}_B$$

If we find U_B we would be able to see things in Schrödinger picture:

$$|\psi'\rangle = \underline{U_B}|\psi\rangle \quad : \quad \langle\psi|\,f(\vec{a}'^\dagger, \vec{a}')\,|\psi\rangle = \langle\psi'|\,f(\vec{a}^\dagger, \vec{a})\,|\psi'\rangle$$

General construction of unitary operator U_B

Take $\mathcal{H}$ such that $B = e^{i\mathcal{H}}$

Define: $\hat{U}_B := e^{i\hat{H}_B}$, $\hat{H}_B = \vec{\hat{a}}^\dagger \mathcal{H}\, \vec{\hat{a}}$

Example: Beamsplitter

$$B = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} = e^{i\mathcal{H}}, \quad \mathcal{H} = \varphi \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\hat{H}_B = \varphi \begin{bmatrix} \hat{a}_1^\dagger, \hat{a}_2^\dagger \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix} = i\varphi \left[a_2^\dagger a_1 - a_1^\dagger a_2 \right]$$

$$\hat{U}_B = e^{i\hat{H}_B}$$

We want to show:

$$\hat{U}_B^+ \hat{a}_k \hat{U}_B = \sum_i B_{ki} \hat{a}_i$$

$$e^{-i \vec{\hat{a}}^+ \mathcal{H} \vec{\hat{a}}} a_k e^{i \vec{\hat{a}}^+ \mathcal{H} \vec{\hat{a}}} = \sum_i \left(e^{i\mathcal{H}}\right)_{ki} \hat{a}_i$$

Proof:

consider $\hat{f}_k(\eta) = \left(\hat{U}_B^+\right)^\eta \hat{a}_k \left(\hat{U}_B\right)^\eta$, $\hat{g}_k(\eta) = \sum_i \left(B^\eta\right)_{ki} \hat{a}_i$

obviously:

$$\hat{f}_k(0) = \hat{g}_k(0) \quad , \quad \| \quad e^{-i\eta \vec{\hat{a}}^+ \mathcal{H} \vec{\hat{a}}}$$

Let us calculate:

$$\hat{f}_k(\eta)' = -i e^{-i\eta \vec{\hat{a}}^+ \mathcal{H} \vec{\hat{a}}} \underbrace{\left[\vec{\hat{a}}^+ \mathcal{H} \vec{\hat{a}}, \hat{a}_k\right]}_{\left\{\left\{ \frac{d\hat{f}_k(\eta)}{d\eta} \right\}\right\}} e^{i\eta \vec{\hat{a}}^+ \mathcal{H} \vec{\hat{a}}} =$$

$$\left[\sum_{ij} \hat{a}_i^+ \mathcal{H}_{ij} \hat{a}_j, \hat{a}_k\right] = \sum_{ij} \mathcal{H}_{ij} \underbrace{\left[\hat{a}_i^+, \hat{a}_k\right]}_{-\delta_{ik}} \cdot \hat{a}_j$$

$\{\{ [AB, C] = A[B,C] + [A,C]B \}\}$

$$= -\sum_j \mathcal{H}_{kj} \hat{a}_j$$

$$= -i \sum_j \mathcal{H}_{kj} \underbrace{e^{-i\eta \vec{\hat{a}}^+ \mathcal{H} \vec{\hat{a}}} \hat{a}_j e^{i\eta \vec{\hat{a}}^+ \mathcal{H} \vec{\hat{a}}}}_{\hat{f}_j(\eta)}$$

$$\hat{f}_k'(\eta) = -i \sum_j \mathcal{H}_{kj} \hat{f}_j(\eta), \qquad \vec{\hat{f}}'(\eta) = -i \mathcal{H} \vec{\hat{f}}(\eta)$$

$$\hat{g}_k(\eta) = \sum_i \left(e^{i\eta\mathcal{H}}\right)_{ki} \hat{a}_i \qquad \hat{g}_k'(\eta) = i \sum_j \mathcal{H}_{kj} \hat{g}_j(\eta)$$

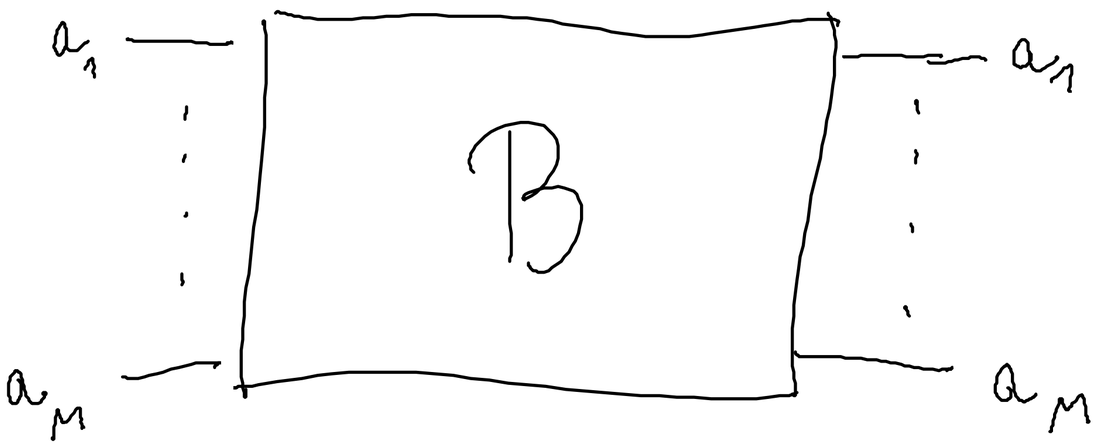
$$\vec{\hat{g}}'(\eta) = i \mathcal{H} \vec{\hat{g}}(\eta) \Longrightarrow \vec{\hat{f}}(\eta) = \vec{\hat{g}}(\eta)$$

$\eta = 1$ ▨

If we have $\hat{U}_B$ we can now move to the Schrödinger picture

Schrödinger picture evolution,

$$\left\{ \begin{array}{l} \vec{\hat{a}}^{\dagger} \neq \vec{\hat{a}}^{\dagger} \\ [\vdots] \neq [\dots\dots] \end{array} \right.$$



$|\psi\rangle$ $\quad$ $|\psi'\rangle$

$$|\psi'\rangle = \hat{U}_B |\psi\rangle, \qquad \hat{U}_B = e^{i \vec{\hat{a}}^{\dagger} \gamma \vec{\hat{a}}}$$

Let $|\psi\rangle = f(\vec{\hat{a}}, \vec{\hat{a}}^{\dagger})|0\rangle$

e.g. $|\psi\rangle = \sum_{n_1 \dots n_M} c_{n_1,\dots,n_M} |n_1,\dots,n_M\rangle = \sum_{n_1 \dots n_M} \frac{(a_1^{\dagger})^{n_1}}{\sqrt{n_1!}} \dots \frac{(a_M^{\dagger})^{n_M}}{\sqrt{n_M!}} |0\rangle$

$$|\psi'\rangle = \hat{U}_B f(\vec{\hat{a}}, \vec{\hat{a}}^{\dagger})|0\rangle =$$

$$= \hat{U}_B f(\vec{\hat{a}}, \vec{\hat{a}}^{\dagger}) \hat{U}_B^{\dagger} \hat{U}_B |0\rangle =$$

$\hat{U} f(\hat{A}) \hat{U}^{\dagger} = \hat{U} \sum_n a_n \hat{A}^n \hat{U}^{\dagger} =$

$\{ \hat{U} \hat{A}^n \hat{U}^{\dagger} = (\hat{U} \hat{A} \hat{U}^{\dagger})^n$

$= \sum_n a_n (\hat{U} \hat{A} \hat{U}^{\dagger})^n =$

$= f(\hat{U} \hat{A} \hat{U}^{\dagger})$

Vaccum invariant under $\hat{U}_B$! $\quad e^{i \vec{\hat{a}}^{\dagger} \gamma \vec{\hat{a}}} |0\rangle = |0\rangle$

$$= \hat{U}_B f(\vec{\hat{a}}, \vec{\hat{a}}^{\dagger}) \hat{U}_B^{\dagger} |0\rangle = f(\hat{U}_B \vec{\hat{a}} \hat{U}_B^{\dagger}, \hat{U}_B \vec{\hat{a}}^{\dagger} \hat{U}_B^{\dagger}) |0\rangle$$

$$\left\{ \hat{U}_B^{\dagger} \vec{\hat{a}} \hat{U}_B = B \vec{\hat{a}} \right. \Rightarrow \hat{U}_B \vec{\hat{a}} \hat{U}_B^{\dagger} = B^{\dagger} \vec{\hat{a}} \quad *$$

$$\hat{U}_B \vec{\hat{a}}^{\dagger} \hat{U}_B^{\dagger} = \vec{\hat{a}}^{\dagger} B$$

$$\left\{ \Rightarrow B \to B^{\dagger} \Rightarrow \gamma \to -\gamma \Rightarrow \right.$$
$$\Rightarrow H_B \to -H_B \Rightarrow \hat{U}_B^{\dagger} \to \hat{U}_B$$

$$= f(B^{\dagger} \vec{\hat{a}}, \vec{\hat{a}}^{\dagger} B)|0\rangle$$

Evolution: $\vec{\hat{a}}^{\dagger} \to \vec{\hat{a}}^{\dagger} B$!

$\vec{\hat{a}} \to B^{\dagger} \vec{\hat{a}}$

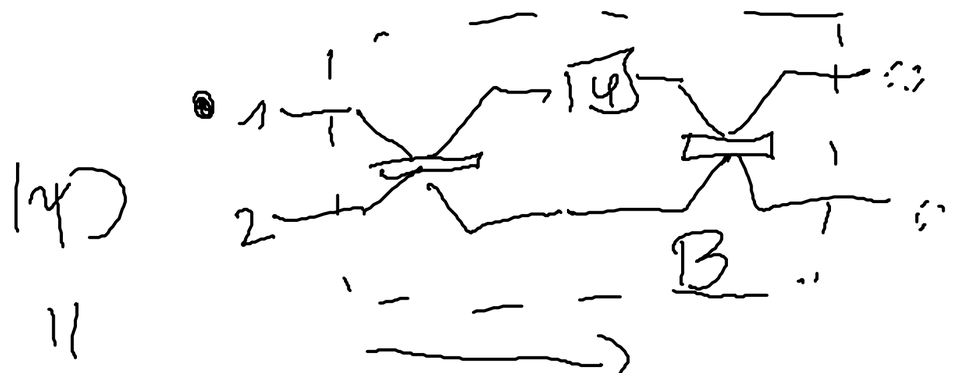
We described passive transformation. (Active transformation mix $\hat{a}$ with $\hat{a}^\dagger$ — later...)

$$\vec{\hat{a}}^\dagger \to \vec{\hat{a}}^\dagger B$$

$$[\hat{a}_1^\dagger, \hat{a}_2^\dagger]\ B\ ] = [\ \bigcirc\ ,\ldots]$$

Example:

Single photon in an interferometer:



$$|\psi'\rangle = \sum_i \hat{a}_i^\dagger B_{i1} |0\rangle$$

$$|1,0\rangle = a_1^\dagger |0\rangle$$

$$B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{i\varphi} & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \frac{\varphi}{2}$$

$$= e^{i\frac{\varphi}{2}} \begin{bmatrix} i\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \\ -\cos\frac{\varphi}{2} & -i\sin\frac{\varphi}{2} \end{bmatrix}$$

$$|\psi'\rangle = e^{i\frac{\varphi}{2}} \left(i\sin\frac{\varphi}{2}\, \hat{a}_1^\dagger - \cos\frac{\varphi}{2}\, \hat{a}_2^\dagger \right) |0\rangle =$$

$$= i\sin\frac{\varphi}{2}\, |1,0\rangle - \cos\frac{\varphi}{2}\, |0,1\rangle$$

prob. amplitudes of 1 photon being in arm 1 or 2.

$$\begin{cases} [B] \cdot \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} = \varepsilon \cdot \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} \end{cases}$$

Example: Coherent states in linear optical network

$$|\psi\rangle = |\vec{\alpha}\rangle = |\alpha_1\rangle \otimes \ldots \otimes |\alpha_m\rangle \quad \boxed{B} \quad |\psi'\rangle = ?$$

$$|\psi\rangle = e^{\vec{\hat{a}}^\dagger \vec{\alpha} - \vec{\alpha}^\dagger \vec{\hat{a}}} |0\rangle \xrightarrow[\vec{\hat{a}} \to B^\dagger \hat{a}]{\vec{\hat{a}}^\dagger \to \vec{\hat{a}}^\dagger B} e^{\vec{\hat{a}}^\dagger B \vec{\alpha} - \vec{\alpha}^\dagger B^\dagger \hat{a}} |0\rangle =$$

$$= |B\vec{\alpha}\rangle =$$

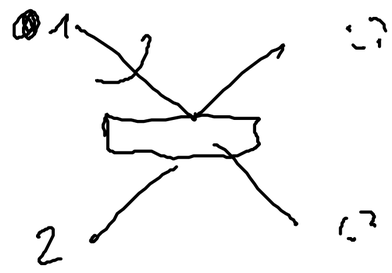
$$= |\textstyle\sum_j B_{1j}\alpha_j\rangle \otimes |\sum_j B_{2j}\alpha_j\rangle \otimes \ldots \otimes |\sum_j B_{mj}\alpha_j\rangle$$

$$\begin{cases} \hat{a}|\alpha\rangle = \alpha|\alpha\rangle \\ |\alpha\rangle = e^{\hat{a}^\dagger \alpha - \alpha^* \hat{a}} |0\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \end{cases}$$

Important difference !

a)

$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$



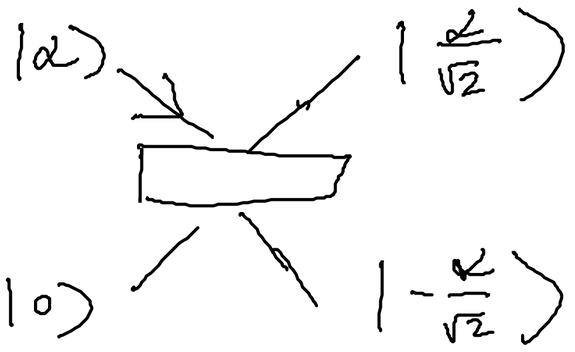
$|\psi\rangle = a_1^\dagger |0\rangle$
$=$
$|1,0\rangle$

$|\psi'\rangle = \frac{1}{\sqrt{2}} \left(|1,0\rangle - |0,1\rangle \right)$

↑ "more entangled state"

b)

$|\psi\rangle = |\alpha\rangle \otimes |0\rangle$



$|\psi'\rangle = \left|\frac{\alpha}{\sqrt{2}}\right\rangle \otimes \left|-\frac{\alpha}{\sqrt{2}}\right\rangle$

↑ "mode product states"

Example Hong-Ou-Mandel interference.

$|\psi\rangle$ $\quad |\psi'\rangle = ?$

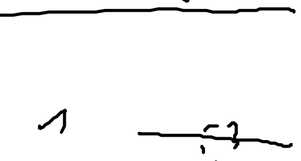
$B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$|\psi\rangle = \hat{a}_1^\dagger \hat{a}_2^\dagger |0\rangle = |1,1\rangle$

$|\psi'\rangle = \frac{1}{\sqrt{2}} (\hat{a}_1^\dagger - \hat{a}_2^\dagger) \frac{1}{\sqrt{2}} (\hat{a}_1^\dagger + \hat{a}_2^\dagger) |0\rangle = \frac{1}{2} \left(\hat{a}_1^{\dagger 2} + \hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_2^\dagger \hat{a}_1^\dagger - \hat{a}_2^{\dagger 2} \right) |0\rangle$

$= \frac{1}{2} \left(\hat{a}_1^{\dagger 2} - \hat{a}_2^{\dagger 2} \right) |0\rangle = \frac{1}{\sqrt{2}} \left(|2,0\rangle - |0,2\rangle \right)$

Example Imperfect interference (eg. imperfect temporal overlap)



$|\psi\rangle = (c_1 b_1^\dagger + c_2 b_2^\dagger) |0\rangle$

$\hat{b}_1 = \int d\omega \; \tilde{f}_1(\omega) \; \hat{a}_1(\omega)$ $\qquad \hat{b}_2 = \int d\omega \; \tilde{f}_2(\omega) \; \hat{a}_2(\omega)$

amn beamsplitter $\begin{pmatrix} \hat{a}_1'(\omega) \\ \hat{a}_2'(\omega) \end{pmatrix} = B \begin{bmatrix} \hat{a}_1(\omega) \\ \hat{a}_2(\omega) \end{bmatrix}$ $\qquad \begin{pmatrix} \hat{b}_1' \\ \hat{b}_2' \end{pmatrix} = B \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$|\psi\rangle = \left(c_1 \int d\omega\, \tilde{f}_1^*(\omega)\, \hat{a}_1^\dagger(\omega) + c_2 \int d\omega\, \tilde{f}_2^*(\omega)\, \hat{a}_2^\dagger(\omega) \right) |0\rangle$$

$$\longrightarrow |\psi'\rangle = \left(c_1 \int d\omega\, \tilde{f}_1^*(\omega)\, \frac{1}{\sqrt{2}} \left(\hat{a}_1^\dagger(\omega) - \hat{a}_2^\dagger(\omega) \right) + \right.$$

$$\left. + c_2 \int d\omega\, \tilde{f}_2^*(\omega)\, \frac{1}{\sqrt{2}} \left(\hat{a}_1^\dagger(\omega) + \hat{a}_2^\dagger(\omega) \right) \right) |0\rangle =$$

$$= \overbrace{\int d\omega\, \hat{a}_1^\dagger(\omega)\, \frac{1}{\sqrt{2}} \left(c_1 \tilde{f}_1^*(\omega) + c_2 \tilde{f}_2^*(\omega) \right) |0\rangle}^{|q_1\rangle} +$$

$$+ \underbrace{\int d\omega\, \hat{a}_2^\dagger(\omega)\, \frac{1}{\sqrt{2}} \left(c_2 \tilde{f}_2^*(\omega) - c_1 \tilde{f}_1^*(\omega) \right) |0\rangle}_{|q_2\rangle}$$

probability of detecting a photon in the upper arm:

$$p_1 = \langle q_1 | q_1 \rangle =$$

$$\begin{cases} \left[a_1(\omega), a_1^\dagger(\omega') \right] = \\ \qquad = \delta(\omega - \omega') \\ \int d\omega \left| \tilde{f}(\omega) \right|^2 = 1 \end{cases}$$

$$= \frac{1}{2} \left[1 + 2\,\mathrm{Re} \left(c_1^* c_2 \underbrace{\int d\omega\, \tilde{f}_1(\omega)\, \tilde{f}_2^*(\omega)}_{V} \right) \right]$$

if $V = 1$ – perfect overlap of modes (perfect interference).