2. Single photon states

General Fock states $|n_1, n_2, \dots\rangle = \frac{a_1^{+n_1}}{\sqrt{n_1!}} \frac{a_2^{+n_2}}{\sqrt{n_2!}} \dots |0\rangle$

- given number of photons in different modes.

single photon states:

$$|\psi\rangle = a_k^+ |0\rangle$$

if we think of $\hat{a}(\vec{k}, \sigma)$ - corresponding to plane waves.

then: $a^+(k,\sigma)|0\rangle$ — single monochromatic photon in the whole space.

energy: $E = \hbar\omega = \hbar kc$

more realistic single photon states:

$$|\psi\rangle = \int d^3\vec{k} \sum_\sigma \tilde{\psi}(\vec{k}, \sigma) \hat{a}^+(\vec{k}, \sigma) |0\rangle$$

general single photon state — complex amplitudes

$\tilde{\psi}(\vec{k}, \sigma)$ - photon wave function in momentum representation.

$$\langle\psi|\psi\rangle = 1 \Rightarrow \int d^3k \sum_\sigma |\tilde{\psi}(\vec{k}, \sigma)|^2 = 1$$

How to define a photon wave function in the position representation. $\vec{\psi}(\vec{r}, t) = ?$

{ Iwo Bialynicki-Birula review: arxiv: quant-ph/0508202

Define:

$$\vec{\psi}_\sigma(\vec{r}, t) := \langle 0 | \hat{\vec{\psi}}_\sigma(\vec{r}, t) | \psi \rangle =$$

$$= \langle 0 | \hat{\vec{\psi}}^{(+)} | \psi \rangle =$$

$$\hat{\vec{\psi}} = \sqrt{\frac{\varepsilon_0}{2}} \left(\hat{\vec{E}} + i\sigma \cdot c \hat{\vec{B}} \right) =$$

$$= \hat{\vec{\psi}}^{(+)} + \hat{\vec{\psi}}^{(-)}$$

$\hat{a} e^{-i\omega t}$ ... $\hat{a}^+ e^{i\omega t}$

$$\hat{\vec{E}} = \vec{E}^{(+)} + \vec{E}^{(-)}$$

↑ positive freq. part

$$\vec{\psi}_\sigma(\vec{r},t) = \langle 0|\, \hat{\vec{\psi}}^{(+)}_\sigma(\vec{r},t)\,|\psi\rangle =$$

$$= \langle 0| \int d^3\vec{k} \sqrt{\frac{\hbar\omega_k}{(2\pi)^3}}\; \vec{e}(\vec{k},\sigma)\, a(\vec{k},\sigma)\, e^{i(\vec{k}\vec{r}-\omega_k t)} \quad |\psi\rangle =$$

$$= \int d^3k \sqrt{\frac{\hbar\omega_k}{(2\pi)^3}}\; \underbrace{\tilde{\psi}(\vec{k},\sigma)\, \vec{e}(\vec{k},\sigma)}\; e^{i(\vec{k}\vec{r}-\omega_k t)}$$

$$\vec{\psi}(\vec{r},t) = \begin{bmatrix} \vec{\psi}_+(\vec{r},t) \\ \vec{\psi}_-(\vec{r},t \end{bmatrix}$$

Can we regard $\vec{\psi}^\dagger\vec{\psi}$ as probability amplitude for detecting photon at a given position $(\vec{\psi}^\dagger\vec{\psi})(\vec{r},t)$ prob. density?

Calculate "naive scalar product":

$$(\vec{\psi}|\vec{\phi}) := \sum_\sigma \int d^3r\, \vec{\psi}^\dagger_\sigma(\vec{r},t)\, \vec{\phi}_\sigma(\vec{r},t) =$$

$$= \sum_\sigma \int d^3\vec{r} \int d^3k\, d^3k' \sqrt{\frac{\hbar\omega_k}{(2\pi)^3}}\sqrt{\frac{\hbar\omega_{k'}}{(2\pi)^3}}\; \tilde{\psi}^*(\vec{k},\sigma)\, \tilde{\varphi}(\vec{k}',\sigma)\, \vec{e}^{\,\dagger}(\vec{k},\sigma)\vec{e}(\vec{k}',\sigma)\, e^{-i(\vec{k}\vec{r}-\omega_k t)+i\vec{k}'\vec{r}-\omega_{k'}t}$$

$$= \sum_\sigma \int d^3k\, \underbrace{\hbar\omega_k}_{\hbar kc}\, \tilde{\psi}^*(\vec{k},\sigma)\, \varphi(\vec{k},\sigma) \quad \neq \langle\psi|\phi\rangle$$

$$\Big\{\ |\psi\rangle = \sum_\sigma \int d^3k\, \tilde{\psi}(\vec{k},\sigma)\, a^\dagger(\vec{k},\sigma)|0\rangle, \qquad \langle\psi|\varphi\rangle = \sum_\sigma \int d^3k\, \tilde{\psi}^*(\vec{k},\sigma)\, \varphi(\vec{k},\sigma)$$

$$(\vec{\psi}|\vec{\psi}) = \sum_\sigma \int d^3k\, \hbar\omega_k \left|\tilde{\psi}(k,\sigma)\right|^2 = \langle\psi|\hat{H}|\psi\rangle$$

$\hat{=}$ up to vacuum energy

$\vec{\psi}$ ~ "energy density amplitude"

$(\vec{\psi}|\vec{\phi}) \hat{=} \langle\psi|\hat{H}|\phi\rangle$

In order to have a scalar product compatible with $\langle\psi|\varphi\rangle$ we have to multiply $(\vec{\psi}|\vec{\phi})$

$$\langle\vec{\psi}|\vec{\varphi}\rangle := \langle\psi|\varphi\rangle = \sum_\sigma \int d^3k\, \tilde{\psi}^*(k,\sigma)\, \tilde{\varphi}(k,\sigma) =$$

$$= \sum_\sigma \int d^3k\, \tilde{\psi}^*(k,\sigma)\, \vec{e}^{\,+}(\bar{k},\sigma)\, \vec{e}\,(\bar{k},\sigma)\, \tilde{\varphi}(k,\sigma) =$$

$$\begin{cases} \vec{\psi}_\sigma(\bar{r},t) = \int d^3k \sqrt{\frac{\hbar\omega_k}{(2\pi)^3}}\, \tilde{\psi}(\bar{k},\sigma)\, \vec{e}\,(\bar{k},\sigma)\, e^{i(\bar{k}\bar{r}-\omega_k t)} \\ \tilde{\psi}(k,\sigma)\, \vec{e}\,(\bar{k},\sigma) = \sqrt{\frac{1}{\hbar\omega_k (2\pi)^3}} \int d^3r\, \vec{\psi}_\sigma(\vec{r},t)\, e^{-i(\vec{k}\vec{r}-\omega_k t)} \end{cases}$$

$$= \frac{1}{(2\pi)^3 \hbar c} \int d^3\vec{r}\, d^3\vec{r}^{\,1} \sum_\sigma \vec{\psi}^{+}_\sigma(\vec{r},t) \underbrace{\int \frac{d^3k}{|\vec{k}|}\, e^{i\vec{k}(\vec{r}-\vec{r}^{\,1})}}_{\| \; \frac{4\pi}{|\bar{r}-\bar{r}^{\,1}|^2}} \vec{\psi}_\sigma(\vec{r}^{\,1},t) =$$

$$= \frac{1}{\hbar c 2\pi^2} \sum_\sigma \int d^3\bar{r}\, d^3\bar{r}^{\,1} \frac{\vec{\psi}^{+}_\sigma(\bar{r}^{\,1})\, \vec{\psi}_\sigma(\bar{r}^{\,1})}{|\bar{r}-\bar{r}^{\,1}|^2}$$

General mode decomposition (localized spatio-temporal modes)

$$\hat{\vec{E}}^{(+)}(\vec{r},t) = \sum_{j,\sigma} \hat{b}_{j,\sigma}\, \vec{u}_{j,\sigma}(\vec{r},t) \quad \text{- general mode decomposition}$$

$$\hat{b}_{j,\sigma} = \int d^3k\; \tilde{u}_{j,\sigma}(\vec{k})\, \hat{a}(\vec{k},\sigma), \qquad \left[\hat{b}_{j,\sigma}, \hat{b}^{+}_{j',\sigma'}\right] = \delta_{jj'}\delta_{\sigma\sigma'}$$

$$\begin{cases} \int d^3k\; \tilde{u}^*_{j,\sigma}(\vec{k})\, \tilde{u}_{j',\sigma'}(\vec{k}) = \delta_{jj}\delta_{\sigma\sigma'} \quad \text{(orthonormality)} \\ \sum_j \tilde{u}_{j,\sigma}(\vec{k})\, \tilde{u}^*_{j,\sigma}(\vec{k}') = \delta^{(3)}(\vec{k}-\vec{k}') \quad \text{(completness)} \end{cases}$$

$\tilde{u}_j(\vec{k})$ — two indices of a "matrix"

unitarity of $\tilde{u}_j(\vec{k})$

$$\underline{\vec{u}_{j,\sigma}(\vec{r},t)} = i\sqrt{\frac{\hbar c}{2\varepsilon_0}} \int d^3k\, \sqrt{|\vec{k}|}\; \tilde{u}^*_{j,\sigma}(\vec{k})\, \vec{e}\,(\vec{k},\sigma)\, e^{i(\vec{k}\vec{r}-\omega_k t)} =$$

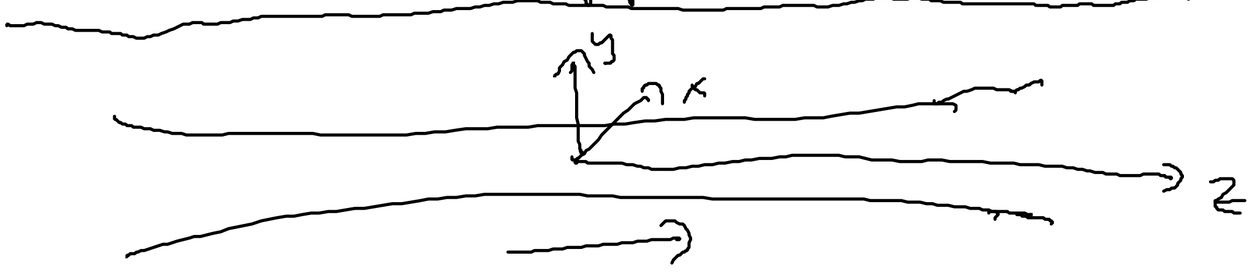
$$= i\sqrt{\frac{\hbar c (2\pi)^3}{2\varepsilon_0}} \int d^3k\, \sqrt{|\vec{k}|}\; \tilde{u}^{+}_{j,\sigma}(\vec{k})\, \frac{1}{\sqrt{(2\pi)^3}}\, \vec{e}\,(k,\sigma)\, e^{i(\vec{k}\vec{r}-\omega_k t)}$$

$$\vec{V}_{\vec{k},\sigma}(\vec{r},t)$$

Transformation between modes is unitary!

We will often think of: $b^+_{j,\sigma}|0\rangle$ —
— single photon in some physical mode (j,σ).

Paraxial approximation



$k_z \gg k_x, k_y$

$$k_z = \sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2} = \frac{\omega}{c}\sqrt{1 - \frac{k_x^2+k_y^2}{\frac{\omega^2}{c^2}}} \simeq \frac{\omega}{c}\left(1 - \frac{k_x^2+k_y^2}{2\frac{\omega^2}{c^2}}\right) =$$

$$= \frac{\omega}{c} - \frac{c}{2\omega}\left(k_x^2+k_y^2\right)$$

It is convenient to consider the following mode decomposition

$$\hat{\vec{E}}^{(+)}(\vec{r},t) = \sum_{\lambda,j,\sigma} \hat{b}_{\lambda,j,\sigma}\, \vec{u}_{\lambda,j,\sigma}(\vec{r},t)$$

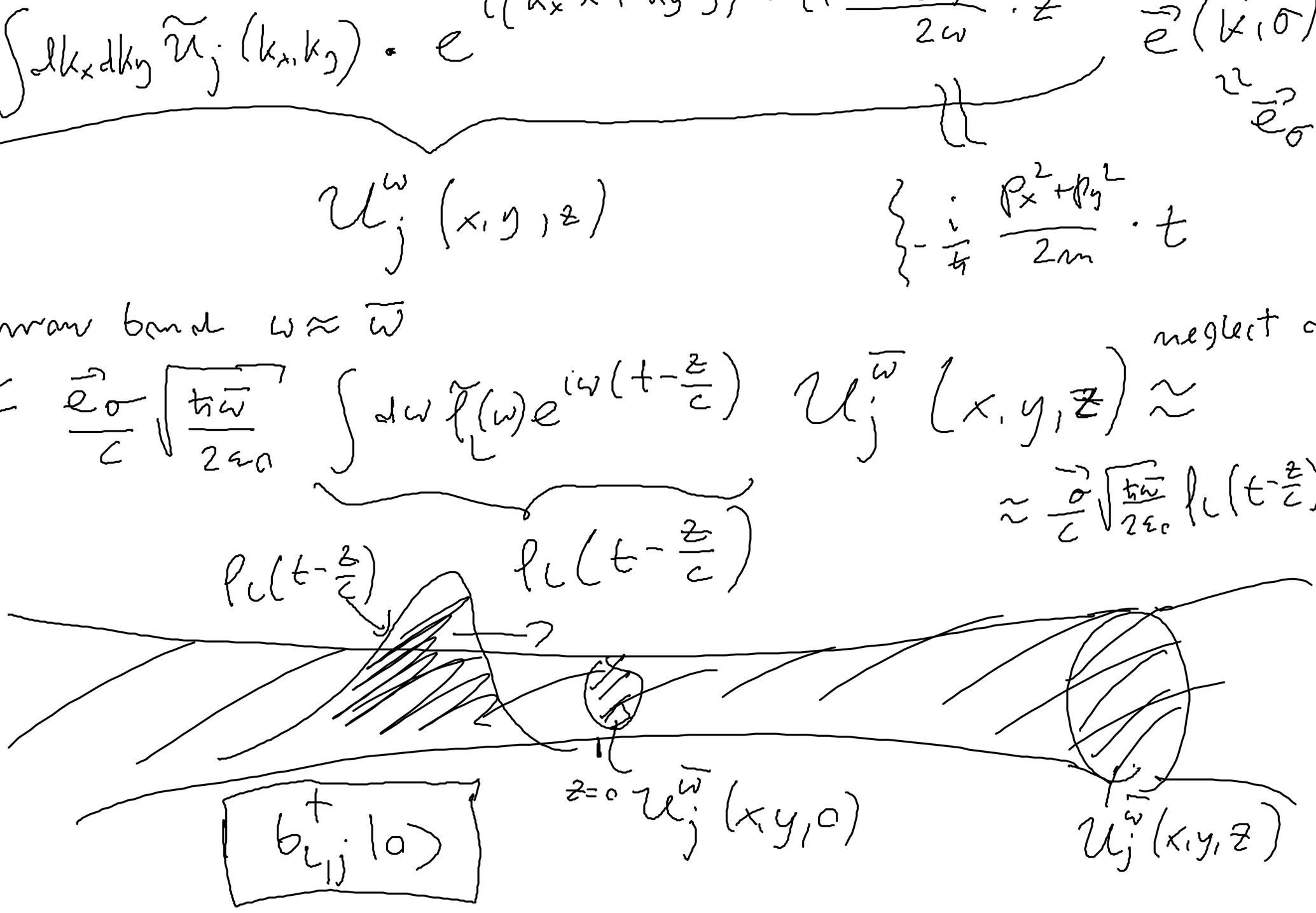
$$\hat{b}_{\lambda,j,\sigma} = \int d\omega\, dk_x\, dk_y\; \tilde{f}_\lambda(\omega)\cdot \tilde{u}_j(k_x,k_y)\; \hat{a}(\vec{k},\sigma)$$

$$\vec{u}_{\lambda,j,\sigma} = \int \frac{d\omega}{c}\sqrt{\frac{\hbar\omega}{2\varepsilon_0}}\, \tilde{f}_\lambda(\omega)\, e^{-i\omega t + i\omega\frac{z}{c}} \cdot$$

$$\underbrace{\int dk_x dk_y\, \tilde{u}_j(k_x,k_y)\cdot e^{i(k_x\cdot x + k_y\cdot y) - i\frac{(k_x^2+k_y^2)c}{2\omega}\cdot z}}_{u^\omega_j(x,y,z)} \cdot \underbrace{\vec{e}(\vec{k},\sigma)}_{\vec{e}_\sigma}$$

$$\left\{ - \frac{i}{\hbar}\frac{p_x^2+p_y^2}{2m}\cdot t \right.$$

narrow band $\omega \approx \bar{\omega}$ — neglect diffraction

$$\simeq \frac{\vec{e}_\sigma}{c}\sqrt{\frac{\hbar\bar{\omega}}{2\varepsilon_0}} \underbrace{\int d\omega\, \tilde{f}_\lambda(\omega) e^{i\omega(t-\frac{z}{c})}}_{f_\lambda(t-\frac{z}{c})}\; u_j^{\bar{\omega}}(x,y,z) \simeq$$

$$\simeq \frac{\vec{e}_\sigma}{c}\sqrt{\frac{\hbar\bar{\omega}}{2\varepsilon_0}}\, f_\lambda\left(t-\frac{z}{c}\right) u_j^{\bar{\omega}}(x,y)$$