Canonical Quantization of E-M field

General

Fields $q^i(\vec{r},t) \longrightarrow \underset{\uparrow}{\mathcal{L}}(\varphi^i, \partial_\nu \varphi^i)$

Lagrangian density

$$\begin{cases} L = \int d^3\vec{r}\, \mathcal{L} \\ \dfrac{\partial \mathcal{L}}{\partial \varphi^i} - \partial_\mu \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^i)} = 0 \end{cases}$$

$$\longrightarrow \underset{|}{\mathcal{H}} = \sum_i \pi^i \partial_t \varphi_i - \mathcal{L}, \qquad H = \int d^3\vec{r}\, \mathcal{H}$$

Hamiltonian density

$$\pi^i = \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi^i)}$$

Quantization: $\varphi^i \to \hat{\varphi}^i$, $\pi^i \to \hat{\pi}^i$

$$[\hat{\varphi}^i(\vec{r},t), \hat{\pi}^j(\vec{r}',t)] = i\hbar\, \delta^{(3)}(\vec{r}-\vec{r}')\, \delta^{ij}$$

$$\dot{\hat{\pi}}^i = -\frac{i}{\hbar}[\hat{\pi}^i, \hat{H}] \qquad \dot{\hat{\varphi}}^i = -\frac{i}{\hbar}[\hat{\varphi}^i, \hat{H}]$$

In Electrodynamics

$$A^\mu = \left[\frac{\varphi}{c}, \vec{A}\right]^T$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \qquad \vec{E} = -\vec{\nabla}\varphi - \partial_t \vec{A}$$

$$\Downarrow$$

$$\vec{\nabla}\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Remaining equations: $\vec{\nabla}\vec{E} = 0 \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c^2}\partial_t \vec{E}$

$$\Downarrow$$

$$\Box \vec{A} = 0$$

We should choose gauge

$$A^\mu \to A^\mu - \partial_\mu \chi$$

We choose Coulomb gauge $\vec{\nabla}\vec{A} = 0$, $\overset{\text{no changes}}{\Longrightarrow}$ $\varphi = 0$

Lagrangian density

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F_{\mu\nu} = \begin{bmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & B_z & -B_y \\ \frac{E_y}{c} & -B_z & 0 & B_x \\ \frac{E_z}{c} & B_y & B_x & 0 \end{bmatrix}$$

$$\frac{\partial \mathcal{L}}{\partial A^\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial_\mu A^\nu} = 0$$

$$\Downarrow$$

$$\Box A^\nu = 0$$

$$\begin{cases} A^0 = \frac{\varphi}{c} \\ \partial_{\nu=0} = \frac{1}{c}\partial_t \end{cases}$$

$$\Pi^\nu = \frac{\partial \mathcal{L}}{\partial(\partial_t A_\nu)} = \frac{1}{c^2 \mu_0}\left(\partial^t A^\nu - \partial^\nu \varphi\right)$$

$$\Pi^0 = 0 \quad , \quad \Pi^k = \frac{1}{c^2\mu_0}\left(-\partial^t A^k + \partial^k \varphi\right) = -\frac{E^k}{c^2\mu_0} = -\varepsilon_0 E^k$$

Quantization: $(i,j = 1,2,3)$

$$\left[A^i(\vec{r},t), \Pi^j(\vec{r}',t)\right] \doteq i\hbar\, \delta^{(3)}(\vec{r}-\vec{r}')\, \delta^{ij}$$

problem: $\vec{\nabla}\vec{A} = 0$

$$\partial_i \left[A^i(\vec{r},t), \Pi^j(\vec{r}',t)\right] \neq i\hbar\, \partial_i\, \delta^{(3)}(\vec{r}-\vec{r}')\, \delta^{ij}$$

$$\Downarrow \qquad \neq 0$$

$$0$$

Due to constraints one needs to modify the Poisson bracket.

to be consistent with the constraints.

$$H = \int d^3r\, \mathcal{H}$$

Hamiltonian: $\hat{\mathcal{H}} = \frac{1}{2}\left(\varepsilon_0 \hat{E}^2 + \frac{\hat{B}^2}{\mu_0}\right) = \frac{1}{2}\left(\frac{\hat{\Pi}^2}{\varepsilon_0} + \frac{\left(\vec{\nabla}\times\hat{A}\right)^2}{\mu_0}\right)$

$$\dot{\hat{\Pi}}^k = -\frac{i}{\hbar}\left[\hat{\Pi}^k, \hat{H}\right] \Rightarrow \Box \hat{A} = 0$$

$$(*)\quad \dot{A}^k = -\frac{i}{\hbar}\left[\hat{A}^k, \hat{H}\right] = \frac{\hat{\Pi}^k}{\varepsilon_0} \qquad \vec{\nabla}\vec{A} = 0 \overset{(*)}{\Rightarrow} \vec{\nabla}\hat{\Pi} = 0$$

① We start with same constraints. Compute Poisson brackets of constraints with Hamiltonian and check if same new constraints appear.

$\Rightarrow$ set of constraints $C_1, C_2, C_3, \dots$

② compute $M_{ij} = \{C_i, C_j\}$

③ define Dirac bracket (replacing Poisson bracket)

$$\{A, B\}_D := \{A, B\}_P - \sum_{ij} \{A, C_i\}_P (M^{-1})_{ij} \{C_j, B\}_P$$

$$\begin{cases} \{C_k, B\}_D = \{C_k, B\}_P - \sum_{ij} \underbrace{\{C_k, C_i\}_P}_{M_{kj}} (M^{-1})_{ij} \{C_j, B\}_P = 0 \\ \text{and if} \\ \{C_k, B\}_P = 0 \quad \Rightarrow \quad \{C_k, P\}_D = 0 \end{cases}$$

In our case: $\vec{\nabla}\vec{A}(\vec{r}) = 0 \qquad \vec{\nabla}\vec{\Pi}(\vec{r}) = 0$

$$\vec{C} = \begin{bmatrix} \vec{\nabla}\vec{A}(\vec{r}) \\ \hline \vec{\nabla}\vec{\Pi}(\vec{r}) \end{bmatrix}$$

$$M = \begin{array}{c} \vec{r}' \\ \vec{r} \end{array} \left[\begin{array}{c|c} 0 & \{\vec{\nabla}\vec{A}(\vec{r}), \vec{\nabla}\vec{\Pi}(\vec{r})\} \\ \hline \{\vec{\nabla}\vec{\Pi}(\vec{r}), \vec{\nabla}\vec{A}(\vec{r})\} & 0 \end{array}\right] =: \left[\begin{array}{c|c} 0 & m \\ \hline -m & 0 \end{array}\right]$$

$$m_{\vec{r},\vec{r}'} = \{\vec{\nabla}\vec{A}(\vec{r}), \vec{\nabla}\vec{\Pi}(\vec{r}')\} = \vec{\nabla}_r \circ \vec{\nabla}_{r'} \, \delta^{(3)}(\vec{r} - \vec{r}') =$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} \vec{\nabla}_r \circ \vec{\nabla}_{r'} \, e^{i\vec{k}(\vec{r}-\vec{r}')} = \int \frac{d^3\vec{k}}{(2\pi)^3} \vec{k}^2 \, e^{i\vec{k}(\vec{r}-\vec{r}')}$$

$$M^{-1} = \left[\begin{array}{c|c} 0 & 1 - m^{-1} \\ \hline m^{-1} & 0 \end{array}\right]$$

$$m^{-1}_{\vec r,\vec r'} = \int \frac{d^3\vec k}{(2\pi)^3}\ \frac{1}{\vec k^2}\ e^{-i\vec k(\vec r-\vec r')}$$

$$\left\{\int m^{-1}_{\vec r,\vec r'}\cdot m_{\vec r'\vec r''}\ d^3\vec r' = \int d^3\vec r' \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}\cdot\frac{\vec k^2}{\vec k'^2}\, e^{i\vec k'(\vec r-\vec r')}\cdot e^{i\vec k(\vec r'-\vec r'')}\right.$$
$$\left. = \delta^{(3)}(\vec r-\vec r'')\right.$$

$$\{A^i(\vec r),\ \Pi^j(\vec r')\}_D = \delta^{ij}\delta^{(3)}(\vec r-\vec r') +$$
$$+\int d^3\vec r''\int d^3\vec r'''\{A^i(\vec r''),\ \vec\nabla\vec\Pi(\vec r'')\}\ (m^{-1})_{\vec r'',\vec r'''}\{\vec\nabla\vec A(\vec r'''),\ \Pi^j(\vec r')\} =$$
$$= \delta^{ij}\delta^{(3)}(\vec r-\vec r') + \int d^3\vec r''\int d^3\vec r'''\int\frac{d^3\vec k}{(2\pi)^3}\ \underbrace{\partial''_i\delta^{(3)}(\vec r-\vec r'')}\ \frac{1}{\vec k^2}\ e^{-i\vec k(\vec r''-\vec r''')}\cdot$$
$$\cdot\underbrace{(-\partial'''_j)\delta^{(3)}(\vec r''-\vec r''')}\cdot\delta^{(3)}(\vec r'''-\vec r') =$$
$$= \delta^{ij}\delta^{(3)}(\vec r-\vec r') + \int\frac{d^3k}{(2\pi)^3}\frac{1}{\vec k^2}\partial_i\partial_j e^{i\vec k(\vec r-\vec r')} = \boxed{-\partial''_i\,\delta^{(3)}(\vec r''-\vec r')}$$

$$= \int\frac{d^3\vec k}{(2\pi)^3}\underbrace{\left(\delta^{ij}-\frac{k_ik_j}{\vec k^2}\right)}e^{i\vec k(\vec r-\vec r')} = :\delta^{tr}_{ij}{}^{(3)}(\vec r-\vec r')$$

$$\left\{\begin{array}{l} \vec k,\ \vec e_2,\ \vec e_1 \\ \vec e_1\vec e_1^{\,T}+\vec e_2\vec e_2^{\,T}+\dfrac{\vec k}{k}\dfrac{\vec k^{\,T}}{k} = \mathbb{1} \\ (\ \text{—}\,\mathbb{1}\,\text{—}\ )_{ij} = \delta_{ij} \\ \left(\vec e_1\vec e_1^{\,T}+\vec e_2\vec e_2^{\,T}\right)^{ij} = \left(\mathbb{1}-\dfrac{\vec k\vec k^{\,T}}{\vec k^2}\right)^{ij} = \delta_{ij}-\dfrac{k_ik_j}{\vec k^2} \end{array}\right.$$

Quantization procedure: $\left[\hat A_i(\vec r),\ \hat\Pi_j(\vec r')\right] = i\hbar\,\delta^{tr}_{ij}(\vec r-\vec r')$

$$[\hat{A}_i(\vec{r}), \hat{\Pi}_j(\vec{r}')] = i\hbar\, \delta_{ij}^{tr}(\vec{r}'-\vec{r}) \quad (*), \qquad \vec{\Pi} = \dot{\vec{A}}\,\varepsilon_0$$

$$[\hat{A}_i(\vec{r}), \hat{\dot{A}}_j(\vec{r}')] = \frac{i\hbar}{\varepsilon_0}\, \delta_{ij}^{tr}(\vec{r}-\vec{r}')$$

$$\Box\vec{A} = 0, \quad \vec{\nabla}\vec{A} = 0 \qquad\qquad \vec{e}(\vec{k},\sigma)\cdot\vec{k} = 0$$

$$\Downarrow$$

$$\hat{\vec{A}}(\vec{r}) = \int d^3\vec{k}\; c(\vec{k}) \sum_{\sigma=1,2} \vec{e}(\vec{k},\sigma)\left[\hat{a}(\vec{k},\sigma)e^{i(\vec{k}\vec{r}-\omega t)} - \hat{a}^{\dagger}(\vec{k},\sigma)e^{-i(\vec{k}\vec{r}-\omega t)}\right]$$

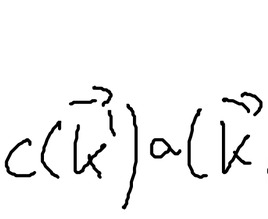
$$\dot{\hat{\vec{A}}}(\vec{r}) = -\int d^3\vec{k}\; c(\vec{k})\,\omega \sum_{\sigma=1,2} \vec{e}(\vec{k},\sigma)\left[\hat{a}(\vec{k},\sigma)e^{i(\vec{k}\vec{r}-\omega t)} + \hat{a}^{\dagger}(\vec{k},\sigma)e^{-i(\vec{k}\vec{r}-\omega t)}\right]$$

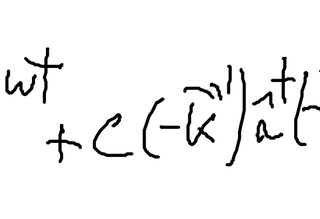
We use $(*)$ to find operator conditions on $\hat{a}(\vec{k},\sigma)$

$$\int d^3\vec{r}\; \vec{A}(\vec{r})\cdot\vec{e}(\vec{k}',\sigma')e^{-i\vec{k}'\cdot\vec{r}} = i(2\pi)^3\left(c(\vec{k}')\hat{a}(\vec{k}',\sigma')e^{i\omega t} - c(-\vec{k}')\hat{a}^{\dagger}(-\vec{k}',\sigma')e^{i\omega t}\right)$$

$(-1)^{\sigma}$

$$\begin{cases} \vec{e}(\vec{k}',\sigma')\cdot\vec{e}(\vec{k}',\sigma) = \delta_{\sigma\sigma'} \\ \vec{e}(\vec{k}',\sigma')\cdot\vec{e}(-\vec{k}',\sigma') = (-1)^{\sigma}\delta_{\sigma\sigma'} \end{cases}$$





$$\int d^3\vec{r}\; \dot{\hat{\vec{A}}}\, e^{-i\vec{k}'\vec{r}}\, \vec{e}(\vec{k}',\sigma') = -i\omega'(2\pi)^3\left(c(\vec{k}')a(\vec{k}',\sigma')e^{-i\omega t} + c(-\vec{k}')\hat{a}^{\dagger}(-\vec{k}',\sigma')e^{+i\omega t}\right)$$

$$\Downarrow$$

$$\hat{a}(\vec{k},\sigma) = -\int \frac{d^3\vec{r}}{(2\pi)^3}\, \frac{e^{-i(\vec{k}\vec{r}-\omega t)}}{2c(\vec{k})}\, \vec{e}(\vec{k},\sigma)\left(i\dot{\vec{A}}(\vec{r}) + \frac{\ddot{\vec{A}}(\vec{r})}{\omega}\right),$$

$$\hat{a}^{\dagger}(\vec{k},\sigma) = \quad - \text{ h.c } -$$

$$[\hat{a}(\vec{k},\sigma), \hat{a}^{\dagger}(\vec{k}',\sigma')] = \int \frac{d^3\vec{r}\, d^3\vec{r}'}{(2\pi)^6\, c(\vec{k})c(\vec{k}')\,4}\, e^{-i(\vec{k}\vec{r}-\omega t)+i(\vec{k}'\vec{r}'-i\omega' t)}.$$

$$2\cdot\sum_{ij}\vec{e}(\vec{k},\sigma)_i\, \frac{\hbar}{\omega\varepsilon_0}\, \underbrace{\delta^{tr}(\vec{r}-\vec{r}')_{ij}}\cdot\, \vec{e}_j(\vec{k}',\sigma') =$$

$$\int \frac{d^3\vec{k}''}{(2\pi)^3}\, e^{i\vec{k}''(\vec{r}-\vec{r}')}\left(\delta_{ij} - \frac{k''_i k''_j}{\vec{k}^2}\right)$$

$$\left[\hat{a}(\vec{k},\sigma),\hat{a}^{+}(\vec{k}',\sigma')\right]=\int\frac{d^3\vec{r}\,d^3\vec{r}'}{(2\pi)^6\,c(\vec{k})\,c(\vec{k}')}\,e^{-i(\vec{k}\vec{r}-\omega t)+i(\vec{k}'\vec{r}'-\omega' t)}\cdot$$

$$2\cdot\sum_{ij}\vec{e}(\vec{k},\sigma)_i\,\frac{\hbar}{\omega\varepsilon_0}\,\underbrace{\delta^{tr}(\vec{r}-\vec{r}')_{ij}}\;\vec{e}_j(\vec{k}',\sigma')=$$

$$\int\frac{d^3k''}{(2\pi)^3}\,e^{i\vec{k}''(\vec{r}-\vec{r}')}\left(\delta_{ij}-\frac{k_i''k_j''}{k^2}\right)$$

$$=\int\frac{d^3k''}{(2\pi)^3}\,\frac{\delta^{(3)}(\vec{k}''-\vec{k})\cdot\delta^{(3)}(\vec{k}''-k')}{2\cdot c(\vec{k})\,c(\vec{k}')}\,\frac{\hbar}{\omega\varepsilon_0}\,e(\vec{k},\sigma)_i\left(\delta_{ij}-\frac{k_i''k_j''}{|\vec{k}|''}\right)=$$

$$=\frac{\hbar}{2\omega\varepsilon_0\,c(\vec{k})^2}\,\delta^{(3)}(\vec{k}-\vec{k}')\;\vec{e}(\vec{k},\sigma)_i\left(\delta_{ij}-\frac{k_ik_j}{|\vec{k}|^2}\right)\vec{e}(\vec{k}',\sigma)_j$$

since $\vec{e}(\vec{k},\sigma)\circ\vec{k}=0$

then $\vec{e}_i\,\frac{k_ik_j}{|k|^2}\,\vec{e}_j=0$

$$=\frac{\hbar}{2\omega\varepsilon_0(2\pi)^3\,c(\vec{k})^2}\,\delta^{(3)}(\vec{k}-\vec{k}')\,\delta_{\sigma,\sigma'}$$

We choose $c(\vec{k})=\sqrt{\frac{\hbar}{(2\pi)^3\,2\omega\varepsilon_0}}$

$$=\delta^{(3)}(\vec{k}-\vec{k}')\,\delta_{\sigma,\sigma'}$$

$$\vec{E}=-\dot{\vec{A}}=\int d^3\vec{k}\sqrt{\frac{\hbar\omega}{(2\pi)^3 2\varepsilon_0}}\sum_{\sigma=1,2}\vec{e}(\vec{k},\sigma)\left(\hat{a}(\vec{k},\sigma)e^{i(\vec{k}\vec{r}-\omega t)}+\hat{a}^{+}(\vec{k},\sigma)e^{-i(\vec{k}\vec{r}-\omega t)}\right)$$

$$\vec{B}=\vec{\nabla}\times\vec{A}=\int d^3\vec{k}\sqrt{\frac{\hbar}{(2\pi)^3 2\varepsilon_0\omega}}\sum_{\sigma=1,2}\vec{k}\times\vec{e}(\vec{k},\sigma)\left(\hat{a}(\vec{k},\sigma)e^{i(\vec{k}\vec{r}-\omega t)}+\hat{a}^{+}(\vec{k},\sigma)e^{-i(\vec{k}\vec{r}-\omega t)}\right).$$

If we move to circular polarization basis:

$$\vec{e}(\vec{k},\pm) = \frac{\vec{e}(\vec{k},1) \pm i\vec{e}(\vec{k},2)}{\sqrt{2}}, \quad \pm \text{ - helicities}$$

$$\vec{E}(\vec{r},t) = \int d^3\vec{k} \sqrt{\frac{\hbar\omega_k}{(2\pi)^3 2\varepsilon_0}} \sum_{\sigma=\pm} \vec{e}(\vec{k},\sigma)\,\hat{a}(\vec{k},\sigma)\,e^{i(\vec{k}\vec{r}-\omega_k t)} + h.c$$

$$\begin{cases} \vec{e}(\vec{k},\pm)^* = \vec{e}(\vec{k},\mp) \end{cases}$$

$$\vec{B}(\vec{r},t) = \int d^3\vec{k} \sqrt{\frac{\hbar\omega_k}{(2\pi)^3 2\varepsilon_0}} \frac{\vec{k}}{c|\vec{k}|} \times \vec{e}(\vec{k},\sigma)\,\hat{a}(\vec{k},\sigma)\,e^{i(\vec{k}\vec{r}-\omega_k t)} + h.c$$

$$\begin{cases} \text{k} \uparrow \quad e(2) \quad e(1) \\ \frac{\vec{k}}{|k|} \times e(1) = e(2), \quad \frac{\vec{k}}{k} \times e(2) = -e(1) \\ \frac{\vec{k}}{k} \times \left(\frac{e(1)+ie(2)}{\sqrt{2}}\right) = \frac{e(2)-ie(1)}{\sqrt{2}} = -i\frac{e(1)+ie(2)}{\sqrt{2}} = -i\,e(\vec{k},+) \\ \sigma = \pm \\ \frac{\vec{k}}{|k|} \times \vec{e}(\vec{k},\sigma) = -i\sigma\,\vec{e}(\vec{k},\sigma) \end{cases}$$

Define the following operator:

$$\hat{\vec{\Psi}}_\sigma(\vec{r},t) = \sqrt{\frac{\varepsilon_0}{2}}\left(\hat{\vec{E}}_\sigma(\vec{r},t) + i\sigma\cdot c\,\hat{\vec{B}}_\sigma(\vec{r},t)\right)^{\wedge}_{\hat{D}} =$$

$$= \int d^3k \sqrt{\frac{\hbar\omega_k}{(2\pi)^3}}\,\vec{e}(\vec{k},\sigma)\hat{a}(\vec{k},\sigma)e^{i(\vec{k}\vec{r}-\omega_k t)}$$

{(In classical electrodynamics: Riman-Silberstein vector)

$$i\hbar\frac{\partial}{\partial t}\hat{\vec{\Psi}}_\sigma(\vec{r},t) = i\hbar\sqrt{\frac{\varepsilon}{2}}\left(c^2\vec{\nabla}\times\hat{\vec{B}}_\sigma(\vec{r},t) - i\sigma c\,\vec{\nabla}\times\hat{\vec{E}}\right) =$$

Maxwell eq.

$$= \hbar c\sigma\,\vec{\nabla}\times\hat{\vec{\Psi}}_\sigma =$$

$$= -i\hbar c\sigma(\vec{S}\circ\vec{\nabla})\hat{\vec{\Psi}}_\sigma$$

$$\begin{cases} \vec{A}\times\vec{B} = -i(\vec{A}\circ\vec{S})\cdot\vec{B} \\ \vec{S} = \begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix}, \quad S_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \\ S_y = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \quad S_z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{cases}$$

The full object describing e-m field

$$\hat{\vec{\Psi}} = \begin{bmatrix} \hat{\vec{\Psi}}_+ \\ \hat{\vec{\Psi}}_- \end{bmatrix} \quad - 6 \text{ component of vector.}$$