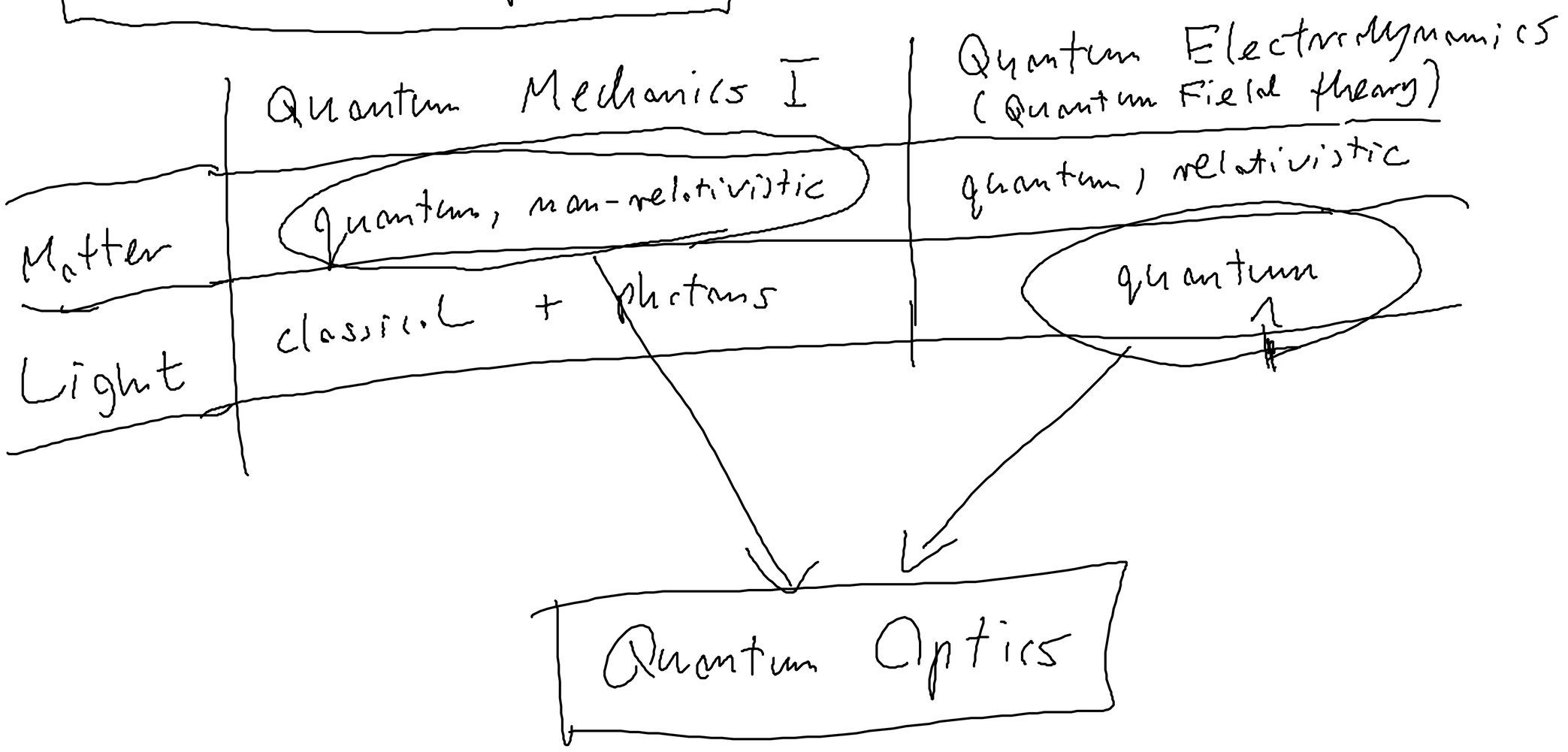
Quantum Optics



I. Quantum description of light

II. Light - Matter interactions

III. Applications (2 lectures delivered by Farid Khalili)

Books: Gerry, Knight "Introductory Quantum Optics"

Scully, Zubairy "Quantum Optics"

I. Quantum description of light

1. Quantization of E-M field

Maxwell equations (no sources)

$$\vec{\nabla}\vec{E}=0 \qquad \vec{\nabla}\times\vec{B}=\frac{1}{c^2}\dot{\vec{E}} \Rightarrow \vec{\nabla}\times\dot{\vec{B}}=\frac{1}{c^2}\ddot{\vec{E}}$$

$$\vec{\nabla}\vec{B}=0 \qquad \vec{\nabla}\times\vec{E}=-\dot{\vec{B}}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) \qquad \frac{1}{c^2}\ddot{\vec{E}}=0 \Rightarrow \vec{\nabla}^2\vec{E}-\frac{1}{c^2}\ddot{\vec{E}}=0$$

$$\vec{\nabla}(\underbrace{\vec{\nabla}\vec{E}}_{0})-\vec{\nabla}^2\vec{E} \qquad \Box\vec{E}=0 \quad \text{wave eq.} + \vec{\nabla}\vec{E}=0$$

$$\vec{E}(\vec{r},t)=\sum_{\vec{k},\sigma=1,2}\vec{e}_{\vec{k},\sigma}\,A_{\vec{k},\sigma}\,e^{i(\vec{k}\vec{r}-\omega_k t)}+c.c$$

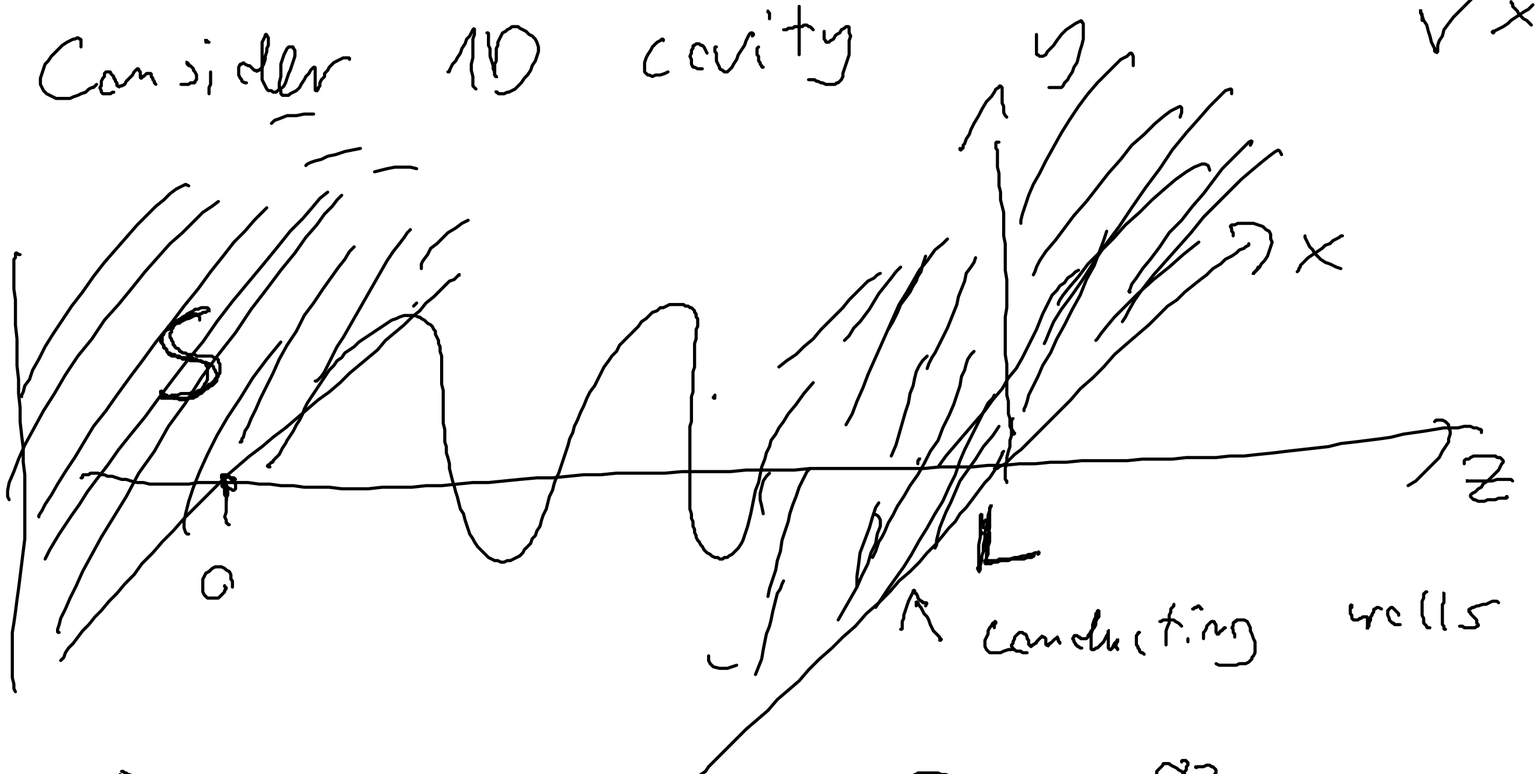
$$\omega_k=c\cdot|\vec{k}| \qquad \vec{\nabla}\vec{E}=0 \Rightarrow \vec{k}\cdot\vec{e}_{\vec{k},\sigma}=0, \quad \vec{e}_{\vec{k},1}\cdot\vec{e}_{\vec{k},2}=0$$

$$\vec{B}(\vec{r},t)=\sum_{\vec{k},\sigma=1,2}\frac{1}{\omega_k}\vec{k}\times\vec{e}_{\vec{k},\sigma}\,e^{i(\vec{k}\vec{r}-\omega_k t)}+c.c$$

Poor physicist approach

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \dot{\vec{E}} \quad (*)$$

Consider 1D cavity



$$\vec{E}(\vec{r},t) = \mathcal{E}(z,t)\,\vec{e}_x = \sum_{j=1}^{\infty} A_j\, q_j(t)\, \sin(k_j \cdot z)\, \hat{e}_x$$

$$k_j = \frac{j\pi}{L}$$

$$\vec{B}(\vec{r},t) = \sum_{j=1}^{\infty} A_j \left(\frac{\dot{q}_j(t)}{c^2 k_j} \right) \cos(k_j \cdot z)\, \hat{e}_y$$

Energy of the field (Hamiltonian)

$$H = \int d^3r\, \frac{1}{2}\left(\varepsilon_0 \vec{E}^2 + \frac{\vec{B}^2}{\mu_0} \right) = \frac{\overbrace{S \cdot L}^{V}}{4} \sum_{j=1}^{\infty} A_j^2 \left(\varepsilon_0 q_j^2 + \frac{1}{c^4 k_j^2 \mu_0} \cdot \dot{q}_j^2 \right)$$

$$= \sum_j \frac{A_j^2 \varepsilon_0 \cdot V}{2\,\omega_j^2}\, \frac{1}{2}\left(\omega_j^2 q_j^2 + \dot{q}_j^2 \right) = \quad \{ \omega_j = c \cdot |k_j|$$

$$\begin{cases} A_j = \sqrt{\dfrac{2\omega_j^2 \cdot m_j}{V \cdot \varepsilon_0}} \\ \omega_j = j\dfrac{\pi \cdot c}{L} \end{cases}$$

$$= \sum_j \frac{1}{2}\left(m_j \dot{q}_j^2 + m_j \omega_j^2 q_j^2 \right) = \sum_j \left(\frac{p_j^2}{2m_j} + \frac{1}{2} m_j \omega_j^2 q_j^2 \right)$$

$$\begin{cases} p_j = m_j \cdot \dot{q}_j \end{cases}$$

$$\begin{cases} \vec{\nabla} \times \vec{E} = -\dot{\vec{B}} \Rightarrow \ddot{q}_j(t) = -\omega_j^2 q_j(t) \end{cases}$$

$q_j, p_j \to \hat{q}_j, \hat{p}_j$

In Heisenberg picture

$$\frac{d\hat{A}}{dt} = -\frac{i}{\hbar}[\hat{A}, \hat{H}]$$

$$\frac{dA}{dt} = \{A, H\}$$

$$\begin{cases} \{A,B\} = \sum_i \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}, \\ A(q_i, p_i) \dots \end{cases} \qquad \dot{q}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\{\cdot,\cdot\} \to -\frac{i}{\hbar}[\cdot,\cdot] \Rightarrow \{q_i, p_j\} = \delta_{ij} \Rightarrow$$

$$\Rightarrow -\frac{i}{\hbar}[\hat{q}_i, \hat{p}_j] = \delta_{ij} \Rightarrow [\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$$

$$\hat{H} = \sum_j \left(\frac{\hat{p}_j^2}{2m_j} + \frac{1}{2} m_j \omega_j^2 \hat{q}_j^2 \right)$$

collection of q. harmonic oscillators

$$\hat{a}_j = \frac{1}{\sqrt{2}}\left(\frac{\hat{q}_j}{x_j} + i\hat{p}_j \frac{x_j}{\hbar} \right) \qquad x_j = \sqrt{\frac{\hbar}{m\omega_j}}$$

$$\hat{a}_j^+ = \frac{1}{\sqrt{2}}\left(\frac{\hat{q}_j}{x_j} - i\hat{p}_j \frac{x_j}{\hbar} \right), \qquad [\hat{a}_j, \hat{a}_k^+] = \delta_{jk}$$

$$\hat{H} = \sum_j \hbar\omega_j \left(\hat{a}_j^+ \hat{a}_j + \frac{1}{2} \right),$$

$$\hat{a}_j(t) = e^{\frac{i\hat{H}t}{\hbar}} \hat{a}_j e^{-\frac{i\hat{H}t}{\hbar}}$$

$$\begin{cases} e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \dots \end{cases}$$

$$\hat{a}_j(t) = \hat{a}_j + \frac{it}{\hbar}\underbrace{[\hat{H}, \hat{a}_j]} + \dots = e^{-i\omega_j \cdot t}\hat{a}_j$$

$\hbar\omega_j [\hat{a}_j^+ \hat{a}_j, \hat{a}_j]$

$$\begin{cases} [AB, C] = A[B,C] + [A,C]B \end{cases}$$

$$\hbar\omega_j [\hat{a}_j^+, \hat{a}_j]\hat{a}_j = -\hbar\omega_j \hat{a}_j$$

$$\hat{a}_j(t) = \hat{a}_j e^{-i\omega_j t}$$

$$\hat{q}_j(t) = \frac{\chi_j}{\sqrt{2}}\left(\hat{a}_j(t) + \hat{a}_j^\dagger(t)\right) \qquad \hat{p}_j(t) = \frac{1}{\sqrt{2}}\frac{\hbar}{i\chi_j}\left(\hat{a}_j(t) - \hat{a}_j^\dagger(t)\right)$$

$$\Downarrow \text{ insert into } \hat{\vec{E}}(\vec{r},t)$$

$$\hat{\vec{E}}(\vec{r},t) = \sum_j A_j \frac{\chi_j}{\sqrt{2}}\left(\hat{a}_j e^{-i\omega t} + \hat{a}_j^\dagger e^{i\omega t}\right)\sin(k_j z)\,\vec{e}_x =$$

$$\sqrt{\frac{2\omega_j^2 m_j}{V\varepsilon_0}} \qquad \sqrt{\frac{\hbar}{m_j\omega_j}}$$

$$= \sum_j \sqrt{\frac{\hbar\omega_j}{V\varepsilon_0}}\left(\hat{a}_j e^{-i\omega_j t} + \hat{a}_j^\dagger e^{i\omega_j t}\right)\sin(k_j z)\,\vec{e}_x$$

$$j = 1, 2, \ldots \infty$$

Quantization in a cavity (standing wave modes)

For running waves we do not impose boundary conditions

$$\hat{\vec{E}} = \sum_j \sqrt{\frac{\hbar\omega_j}{V\varepsilon_0}}\Big[\left(\hat{a}_{j,s} e^{-i\omega_j t} + \hat{a}_{j,s}^\dagger e^{i\omega_j t}\right)\sin(k_j z)\,\vec{e}_x$$

$$+ \left(\hat{a}_{j,c} e^{-i\omega_j t} + \hat{a}_{j,c}^\dagger e^{i\omega_j t}\right)\cos(k_j z)\,\vec{e}_x\Big]$$

$$\begin{cases} \hat{a}_{\pm j} = \dfrac{\hat{a}_{j,c} \mp i\,\hat{a}_{j,s}}{\sqrt{2}} \qquad \hat{a}_{-j} = \dfrac{\hat{a}_{j,c} + i\hat{a}_{j,s}}{\sqrt{2}} \qquad \omega_j = \dfrac{|j|\pi c}{L} \\ [\hat{a}_j, \hat{a}_{j'}] = \delta_{jj'} \end{cases}$$

$$= \sum_{\substack{j=-\infty \\ j\neq 0}}^{+\infty} \sqrt{\frac{\hbar\omega_j}{2V\varepsilon_0}}\left(\hat{a}_j e^{i(k_j z - \omega_j t)} + \hat{a}_j^\dagger e^{-i(k_j z - \omega_j t)}\right)\hat{e}_x$$

$\hat{a}_{-10}, \hat{a}_{5}, \ldots$

General 3D form

$$\begin{cases} \hat{\vec{E}}(\vec{r},t) = \sum_{\vec{k},\sigma=1,2} \sqrt{\frac{\hbar\omega_k}{2V\varepsilon_0}}\, \vec{e}_{\vec{k},\sigma}\, \hat{a}_{\vec{k},\sigma}\, e^{i(\vec{k}\vec{r}-\omega_k t)} + h.c \\ \hat{\vec{B}}(\vec{r},t) = \sum_{\vec{k},\sigma=1,2} \sqrt{\frac{\hbar}{2V\omega_k\varepsilon_0}}\, \vec{k}\times\vec{e}_{\vec{k},\sigma}\, \hat{a}_{\vec{k},\sigma}\, e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + h.c \\ [\hat{a}_{\vec{k},\sigma}, \hat{a}^{+}_{\vec{k}',\sigma'}] = \delta_{\vec{k},\vec{k}'}\,\delta_{\sigma,\sigma'} \end{cases}$$

States of EM field with fixed energy.

$\{\hat{a}_{\vec{k},\sigma} \equiv \hat{a}_l \qquad l=1,\dots$

$|n_1\rangle \otimes |n_2\rangle \otimes \dots = |n_1, n_2, \dots\rangle = |\{n\}\rangle$

n_l – how many photons are in mode l

$E_{|\{n\}\rangle} = \sum_l \hbar\omega_l \left(n_l + \frac{1}{2}\right)$, $|0\rangle := |0,0,0\dots\dots\rangle$

Vacuum state

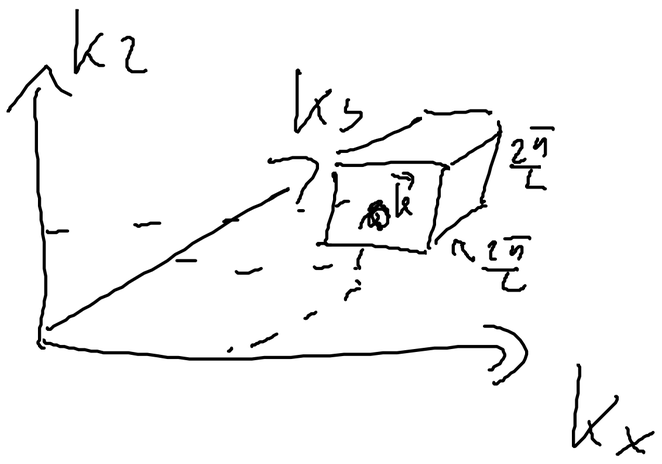
General state:

$$|\psi\rangle = \sum_{\{n\}} c_{\{n\}} |\{n\}\rangle, \quad \sum_{\{n\}} |c_{\{n\}}|^2 = 1$$

Continuous limit:

$$V = L^3 \qquad L \to \infty$$

$$k_x = \frac{2\pi j_x}{L}, \quad k_y = \frac{2\pi j_y}{L}, \quad k_z = \frac{2\pi j_z}{L} \qquad j_i = 1, 2 \dots$$



$$V_{\vec{k}} = \left(\frac{2\pi}{L}\right)^3$$

$$\sum_{\vec{k}} a_{\vec{k}} \longrightarrow c \int d^3k \; a(\vec{k}) \qquad , \qquad \left[a(\vec{k}), a^+(\vec{k})\right] = \delta^{(3)}(\vec{k}-\vec{k}')$$

$$a_{\vec{k}} = c \int_{V_{\vec{k}}} d^3\vec{k}' \, a(\vec{k}') \qquad \left[a_{\vec{k}}, a_{\vec{k}}^+\right] = c^2 \int_{V_{\vec{k}}} d^3\vec{k}' \int_{V_{\vec{k}}} d^3\vec{k}' \; \delta^{(3)}(\vec{k}'-\vec{k}'')$$

$$c^2 \cdot V_{\vec{k}} = 1 \qquad c = \left(\frac{L}{2\pi}\right)^{\frac{3}{2}} = \sqrt{\frac{V}{(2\pi)^3}}$$

$$\sum_{\vec{k},\sigma} \sqrt{\frac{\hbar\omega}{2V\varepsilon_0}} \longrightarrow \int d^3\vec{k} \sum_{\sigma} \sqrt{\frac{\hbar\omega_k}{(2\pi)^3 2\varepsilon_0}}$$

$$\vec{E}(\vec{r},t) = \int d^3\vec{k} \sum_{\sigma} \sqrt{\frac{\hbar\omega_k}{(2\pi)^3 2\varepsilon_0}} \; \vec{e}(\vec{k},\sigma) \cdot \hat{a}(\vec{k},\sigma) \, e^{i(\vec{k}\vec{r} - \omega_k t)} + h.c$$

$$\left[\hat{a}(\vec{k},\sigma), \hat{a}^+(\vec{k}',\sigma')\right] = \delta_{\sigma\sigma'} \, \delta^{(3)}(\vec{k}-\vec{k}')$$