A GUIDE TO READING GIBBS

Statistical mechanics without Planck's constant

How the classical framework works, where its normalization is arbitrary, and what quantum theory adds.

The central answer

Gibbs can assign probabilities and derive many equilibrium relations without a preferred unit of action. The modern factor h^n fixes the scale for counting quantum states. It determines the entropy normalization, while quantum spectra and particle statistics supply further physical information.

A historical distinction. Gibbs's *Elementary Principles in Statistical Mechanics* appeared in 1902, after Planck's 1900 radiation work. Thus "without Planck's constant" describes the assumptions of Gibbs's construction, not the date of his book. Planck's 1901 paper records the 1900 presentations and introduces energy elements proportional to frequency. Gibbs explicitly excludes radiation from the scope of his treatise. [G, p. 41, footnote; P, pp. 553, 561]

1. The classical framework

A mechanical state is a point Γ = (q, p): all generalized positions and their conjugate momenta. Let n be the number of coordinate-momentum pairs and H the Hamiltonian. An *ensemble* is a probability distribution over such states, not an extra physical gas. Gibbs's phase-space measure is [G, p. 6, eq. (12)]

$$d\Gamma = d^n q\, d^n p, \qquad d\mathrm{Prob} = \rho(\Gamma)\, d\Gamma, \qquad \int \rho\, d\Gamma = 1. \tag{1}$$

The density ρ need not be dimensionless. Only its integral over a region is a probability. An observable A has ensemble average

$$\langle A \rangle = \int A(\Gamma)\rho(\Gamma)\, d\Gamma. \tag{2}$$

Hamiltonian flow preserves phase-space volume (Liouville's theorem). This makes dΓ a natural invariant measure. It does not, by itself, select a unique equilibrium distribution or prove that an isolated system approaches equilibrium. [G, pp. 9-10, eqs. (21)-(23) and accompanying text]

For canonical equilibrium, write $\beta = 1/(k_B T)$. Assuming the normalization integral I(β) is finite,

$$\rho(\Gamma) = \frac{e^{-\beta H(\Gamma)}}{I(\beta)}, \qquad I(\beta) = \int e^{-\beta H}\, d\Gamma. \tag{3}$$

This is already a normalized classical probability model. No elementary cell size is needed to calculate its averages. The question of a dimensionless *state count* is a separate step.

2. The scale inside Gibbs's logarithm

Gibbs writes the canonical distribution using ε for energy, Θ for the modulus of distribution, P for probability density, and η for its natural logarithm. His equations on printed p. 33 are

$$\eta = \log P = \frac{\psi - \varepsilon}{\Theta}, \qquad P = e^{(\psi-\varepsilon)/\Theta}. \tag{G (90)-(91)}$$

$$e^{-\psi/\Theta} = \int e^{-\varepsilon/\Theta}\, d\Gamma. \tag{G (92)}$$

In modern thermodynamic units, $\Theta = k_B T$. The numerical value of P is understood in chosen mechanical units. On p. 34 Gibbs expressly notes that ψ depends on the units of energy and time. These equations do not determine a universal quantum of action.

Why units matter

A conjugate product $p_i q_i$ has dimensions of energy times time, or action. Consequently [G, p. 13, paragraph after eq. (30)]

$$[d\Gamma] = [\text{action}]^n, \qquad [\rho] = [\text{action}]^{-n}. \tag{4}$$

A logarithm must act on a dimensionless number. Taking log P therefore means taking the logarithm of the density's numerical value relative to some reference. Gibbs identifies this dependence on p. 19:

> **Gibbs, p. 19, immediately before equation (47)**
>
> "The index of probability is therefore affected by an additive constant when we change our units of time and energy."

Make the reference explicit

Choose any fixed positive action α. Keep it independent of temperature, volume, and external controls. Define a dimensionless phase-space measure and its partition function:

$$d\mu_\alpha = \frac{d\Gamma}{\alpha^n}, \qquad Z_\alpha = \int e^{-\beta H}\, d\mu_\alpha. \tag{5}$$

The density with respect to $d\mu_\alpha$ is $f_\alpha = \rho\alpha^n$. Thus $f_\alpha d\mu_\alpha = \rho d\Gamma$ represents the same probability measure. In this convention the entropy and free energy are

$$S_\alpha = -k_B \int \rho \ln(\rho\alpha^n)\, d\Gamma, \qquad F_\alpha = -k_B T \ln Z_\alpha. \tag{6}$$

For the canonical density, substitute $\ln(\rho\alpha^n) = -\beta H - \ln Z_\alpha$ into the integral. With $U = \langle H \rangle$, this immediately gives

$$S_\alpha = k_B(\ln Z_\alpha + \beta U), \qquad F_\alpha = U - TS_\alpha. \tag{7}$$

The symbol α is our explicit reference action, not a new physical constant. Gibbs's classical construction leaves its value undetermined.

3. What a change of scale changes

Let α and α′ be two fixed reference actions, and keep the system's n degrees of freedom unchanged. From the definition of Z,

$$Z_{\alpha'} = \left(\frac{\alpha}{\alpha'}\right)^n Z_\alpha. \tag{8}$$

The canonical probability measure is exactly the same in either convention:

$$\frac{e^{-\beta H}}{Z_\alpha}\frac{d\Gamma}{\alpha^n} = \frac{e^{-\beta H}\,d\Gamma}{\int e^{-\beta H}\,d\Gamma}. \tag{9}$$

The reference action cancels between numerator and denominator. In contrast, equations (6)-(8) give definite shifts of the thermodynamic potentials:

$$F_{\alpha'} - F_\alpha = nk_{\mathrm{B}}T\ln\left(\frac{\alpha'}{\alpha}\right), \tag{10a}$$

$$S_{\alpha'} - S_\alpha = -nk_{\mathrm{B}}\ln\left(\frac{\alpha'}{\alpha}\right). \tag{10b}$$

This is the additive entropy constant discussed in Gibbs's p. 184 footnote. At fixed composition, it cancels between any two states. It is useful to be precise about which predictions are unaffected:

Quantity	Effect of changing α at fixed n
Normalized averages ⟨A⟩	Unchanged, by equation (9).
Mean energy U = -∂ ln Z / ∂β	Unchanged: the added term in ln Z is independent of β.
Pressure and generalized forces	Unchanged: the scale is independent of the external controls.
Heat capacities and entropy differences	Unchanged at fixed composition.
Entropy S and free energy F	Shift as in equations (10a)-(10b).
Chemical potential μ	Can shift because the number of degrees of freedom varies with particle number.

The connection to thermodynamics

Gibbs's eq. (114), printed p. 44, is $d\langle\varepsilon\rangle = -\Theta\, d\langle\eta\rangle - \Sigma\langle A_i\rangle\, da_i$. In our notation, and at fixed composition, it becomes

$$dU = T\,dS_\alpha - \sum_i X_i\,d\lambda_i. \tag{11}$$

Here λ_i are external controls and X_i their conjugate forces. For volume, X = pressure. Adding a constant to S leaves this relation intact. This explains how classical statistical mechanics supports thermodynamics without selecting α.

Scope: this is an equilibrium argument. The fine-grained Gibbs entropy of an isolated Hamiltonian ensemble is conserved under Liouville evolution; an irreversible entropy increase requires additional reasoning about macroscopic description, coarse graining, or exchange with the surroundings. [G, p. 17, eqs. (43)-(44)]

4. Worked example: a classical ideal gas

Consider N identical, structureless particles of mass m in a volume V, with only translational kinetic energy. Here n = 3N. Count states without retaining arbitrary particle labels, giving the factor 1/N!. Gibbs discusses this generic versus specific counting in Chapter XV. [G, pp. 187-189; p. 203, eq. (546)]

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m}, \qquad Z_N(\alpha) = \frac{1}{N!\,\alpha^{3N}} \int e^{-\beta H}\, d^{3N}q\, d^{3N}p. \tag{12}$$

For this counting convention the measure is $d\Gamma/(N!\alpha^{3N})$. If ρ is written on labeled coordinates, the logarithm in equation (6) becomes $\ln(\rho N!\alpha^{3N})$. Each position integral contributes V. Each of the 3N momentum integrals is Gaussian: $\int \exp[-\beta p^2/(2m)]\, dp = (2\pi m/\beta)^{1/2}$. Therefore

$$Z_N(\alpha) = \frac{1}{N!}\left[\frac{V(2\pi m k_B T)^{3/2}}{\alpha^3}\right]^N. \tag{13}$$

Take derivatives of ln Z while keeping N fixed. The arbitrary action scale disappears:

$$U = -\frac{\partial \ln Z_N}{\partial \beta} = \frac{3}{2} N k_B T, \qquad p = k_B T \left(\frac{\partial \ln Z_N}{\partial V}\right)_{T,N} = \frac{N k_B T}{V}. \tag{14}$$

Hence $pV = Nk_BT$ and $C_V = (3/2)Nk_B$. These successful classical results require no choice of α. The derivation also shows why an overall factor in Z can disappear from one prediction and survive in another.

The entropy retains the scale

Using equation (7) and Stirling's approximation $\ln N! \simeq N \ln N - N$ gives, for large N,

$$S_\alpha \simeq N k_B \left[\ln\left(\frac{V}{N}\frac{(2\pi m k_B T)^{3/2}}{\alpha^3}\right) + \frac{5}{2}\right]. \tag{15}$$

With the quantum choice α = h, this is the Sackur-Tetrode entropy in the classical dilute-gas regime. At a fixed temperature, changing V_1 to V_2 gives $\Delta S = Nk_B \ln(V_2/V_1)$ for every α. Absolute entropy does depend on that choice.

Changing particle number is different

Define $\lambda_\alpha = \alpha/(2\pi m k_B T)^{1/2}$, which has dimensions of length. In the same large-N approximation,

$$\mu_\alpha = \left(\frac{\partial F_\alpha}{\partial N}\right)_{T,V} \simeq k_B T \ln\left(\frac{N\lambda_\alpha^3}{V}\right), \qquad \mu_{\alpha'} - \mu_\alpha = 3 k_B T \ln\left(\frac{\alpha'}{\alpha}\right). \tag{16}$$

The entropy offset is proportional to N, so it is not a single ignorable constant when N varies. Gibbs discusses constraints on entropy constants when the amount of substance varies in the prose on pp. 206-207.

Two distinct counting choices

α^{3N} supplies a unit for phase-space volume. N! removes repeated labelings of identical particles. Fixing one does not determine the other. The quantum gas reduces to this N! prescription in its Maxwell-Boltzmann limit; quantum-degenerate Bose and Fermi gases need their full quantum statistics.

5. Why quantum theory selects h

A simple counting argument fixes the normalization. Take a free particle on a periodic interval of length L. Its momentum eigenstates obey ψ(q + L) = ψ(q). With p = ħk, this gives

$$k_j = \frac{2\pi j}{L}, \qquad p_j = \frac{2\pi\hbar j}{L} = \frac{hj}{L}, \qquad j \in \mathbb{Z}. \tag{17}$$

Adjacent momenta are separated by h/L. An interval Δp containing many levels therefore contains approximately LΔp/h states. For a smooth function g(p),

$$\sum_j g(p_j) \simeq \frac{L}{h}\int g(p)\,dp = \frac{1}{h}\int_0^L dq \int dp\, g(p). \tag{18}$$

Extending the count to n distinguishable canonical degrees of freedom yields the leading semiclassical measure dΓ/h^n. With internal degeneracies omitted, the corresponding quantum trace is approximated by

$$Z_{\mathrm{q}} = \mathrm{Tr}\, e^{-\beta\widehat{H}} \simeq \frac{1}{h^n}\int e^{-\beta H(q,\,p)}\, d\Gamma, \qquad h = 2\pi\hbar. \tag{19}$$

Identical particles require the appropriate quantum statistics; the dilute-gas limit restores the 1/N! factor on page 4. This elementary derivation, adapted to one dimension, follows the state-counting argument in [T, §3.1]; a quantum-trace derivation is given in [T, §2.1.1]. It establishes the scale, including the factor 2π, rather than choosing a convenient unit.

Interpret the phase-space cell carefully

One state per h^n is a semiclassical density-of-states statement. It does not mean that every quantum state occupies a separate rectangular tile in phase space. The uncertainty inequality alone does not derive this count or its exact normalization.

What has been added to Gibbs

The classical probability measure was already well defined. Quantum mechanics now supplies a physical state-counting convention, α = h, for matching the classical approximation to a quantum system. In Gibbs's formulas, the place where this enters is the measure of the integral in eq. (92), and therefore the reference implicit in log P.

The two constants have different jobs. k_B converts temperature to energy and dimensionless entropy to thermodynamic entropy; h has dimensions of action and enters the quantum state count. In Gibbs's Chapter XIV notation, 1/K = k_B. [G, pp. 184-185, especially eq. (495)]

When the classical gas approximation applies

For the gas on the previous page, $\lambda_T = h/(2\pi m k_B T)^{1/2}$ is the thermal wavelength. The nondegenerate regime requires $N\lambda_T^3/V$ to be much smaller than one. For the continuum momentum approximation, the box must also be large compared with λ_T. These conditions explain how h can set the validity of a classical approximation even when it cancels out of the resulting equation of state.

Equation (19) remains an approximation. Quantized energy levels, quantum correlations, and Bose or Fermi statistics can change observable predictions. The next example separates those effects from normalization.

6. An oscillator exposes the limit

For one oscillator of angular frequency ω, the classical Gaussian integral gives

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2, \qquad Z_{\mathrm{cl}}(\alpha) = \frac{2\pi}{\beta\alpha\omega}. \tag{20}$$

Set α = h. The state-count normalization is now fixed, yet equipartition still gives

$$Z_{\mathrm{cl}}(h) = \frac{1}{\beta\hbar\omega}, \qquad U_{\mathrm{cl}} = k_{\mathrm{B}}T, \qquad C_{\mathrm{cl}} = k_{\mathrm{B}}. \tag{21}$$

The modern quantum oscillator has $E_j = \hbar\omega(j + 1/2)$, j = 0, 1, 2, … . Summing a geometric series instead of a classical integral gives [T, §3.4]

$$Z_{\mathrm{q}} = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}}, \qquad U_{\mathrm{q}} = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}. \tag{22}$$

Differentiate U at fixed ω. With $x = \hbar\omega/(k_{\mathrm{B}}T)$,

$$\frac{C_{\mathrm{q}}}{k_{\mathrm{B}}} = \frac{x^2 e^x}{(e^x - 1)^2}. \tag{23}$$

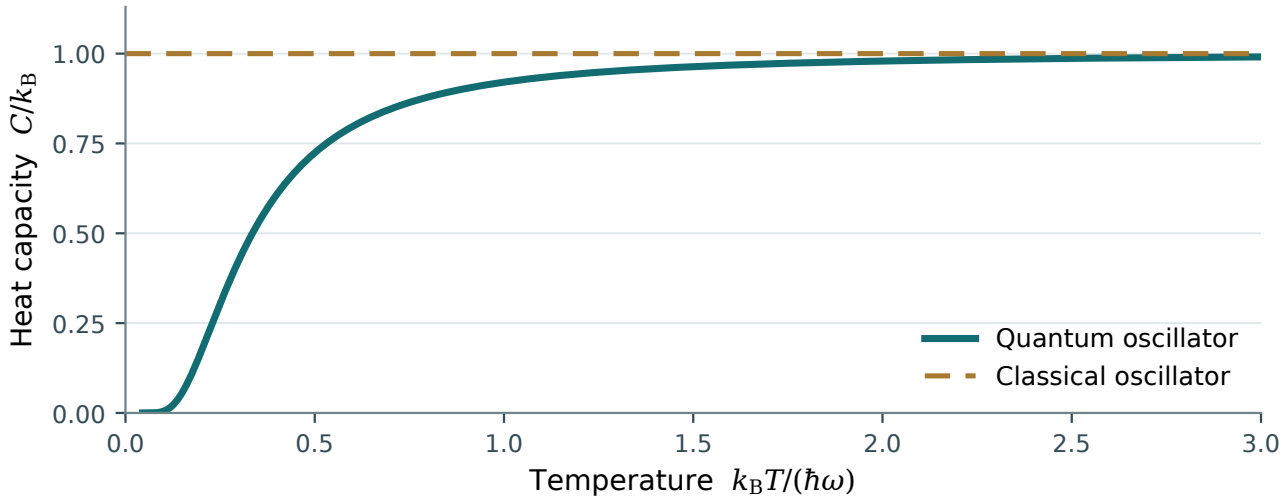


Figure 1. Heat capacities calculated from equations (21) and (23), with ω fixed. The quantum result approaches the classical value when $k_{\mathrm{B}}T$ is large compared with ħω, and tends to zero at low temperature.

The factor 1/h in the classical integral never changes U_{cl} or C_{cl}. The new temperature dependence comes from the **discrete energy spectrum**. At high temperature, the quantum sum approaches the normalized classical integral; at low temperature, it does not.

Check your understanding

Can a different constant α make the classical heat capacity fall to zero? No. Its contribution to ln Z is independent of β, so it disappears from -∂ ln Z/∂β. Changing the spectrum is additional physics.

Historical note: the ħω/2 term is the modern zero-point energy. Planck's thermal oscillator energy is the excitation term with the ground-state energy subtracted. At fixed ω this subtraction does not change the heat capacity. [P, §10]

7. Reading Gibbs with modern notation

Use the book's **printed pages** for citations. The file-page column below is specific to the supplied 240-page Dover DJVU: in this part of the file it is the printed page plus 16. Other scans have different offsets.

Printed / file page	Gibbs location	What to look for
6 / 22	Chapter I, eq. (12)	The element of phase-space volume.
9-10 / 25-26	Eqs. (21)-(23) and text	Density and volume conservation under Hamiltonian evolution.
13 / 29	After eq. (30)	Phase-space volume has dimensions action to the nth power.
16, 18 / 32, 34	Eqs. (38)-(40), (46)	Probability density, log P, and normalization.
19 / 35	Eq. (47)	Unit changes add a constant to the probability index.
33-34 / 49-50	Eqs. (90)-(92), then text	Canonical normalization and the unit dependence of ψ.
44 / 60	Eqs. (114), (116)	Statistical and thermodynamic fundamental relations.
184-185 / 200-201	Footnote; eq. (495)	Entropy differences; the temperature conversion 1/K = k_B.
187-189 / 203-205	Chapter XV	Generic versus specific phases for identical particles.
203, 206 / 219, 222	Eq. (546); later prose	Factorials and the limits on entropy constants when particle number varies.

Notation to watch

We use H for Gibbs's energy ε, ρ for his density P, and S for thermodynamic entropy. His statistical η is log P, so its *negative* average corresponds to entropy. He also reuses η for entropy in the thermodynamic comparison. Our N counts particles; Gibbs's N in Chapter I counts ensemble members. The h that indexes species in Chapter XV is not Planck's constant.

A final distinction

Gibbs's theory determines the form of classical equilibrium probabilities and many relations among macroscopic quantities. It leaves the action scale in the entropy normalization open. Quantum state counting fixes that scale to h; quantum spectra and statistics explain the departures from classical predictions.

Sources and further reading

[G] J. Willard Gibbs, *Elementary Principles in Statistical Mechanics, Developed with Especial Reference to the Rational Foundation of Thermodynamics* (1902), supplied Dover reprint. All page and equation references were checked against the supplied scan. Online text of the original edition.

[P] Max Planck, "Ueber das Gesetz der Energieverteilung im Normalspectrum," *Annalen der Physik* **4**, 553-563 (1901). See the opening note and §10, especially ε = hν on p. 561. Original paper.

[T] David Tong, *Statistical Physics*, University of Cambridge lecture notes. Chapter 2, §§2.1-2.2: classical normalization and gases. Chapter 3, §§3.1 and 3.4: state counting and the quantum oscillator. The derivations and figure in this handout are presented in modern notation.